Univerzita Palackého

Přírodovědecká fakulta

Martin Kuřil

Variety uspořádaných pologrup

Habilitační práce

obor: Algebra a geometrie

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Obsah

Úvod

Habilitační práce je souborem článků [1] – [4]. Tři z nich byly publikovány v časopise Semigroup Forum, který je evidován v databázi Web of Science, jeden byl publikován v časopise Acta Scientiarum Mathematicarum, který je evidován v databázi Scopus.

Práce stručně popisuje hlavní výsledky článků [1] – [4]. Na vhodných místech jsou připomenuty definice a také je stručné popsán kontext našeho výzkumu, tedy práce a výsledky jiných autorů vztahující se ke studované problematice.

Aby habilitační práce byla kompaktnější, sestává z vybraných článků, které se, až na jeden, věnují jednomu tématu – varietám uspořádaných pologrup. Jeden článek se zabývá enumerací uspořádaných pologrup. Po vyslovení definice nějaké struktury (například uspořádané pologrupy) totiž přirozeně vyvstane otázka, kolik takových struktur existuje. V referencích lze pak najít seznam dalších prací autora z teorie pologrup ([5] – [16]).

Dva z článků, zařazených do této práce, mají spoluautora. Spoluatorem článku [1] je Libor Polák a spoluautorem článku [2] je Petr Gajdoš. V prvním případě byl přínos autorů rovnocenný. Výsledků článku [1] bylo dosaženo na základě společných konzultací a diskusí. Ostatně, varietám polosvazově uspořádaných pologrup se autor začal věnovat již ve své diplomové práci [5], jejímž vedoucím byl právě docent Libor Polák. Co se týká článku [2], autor sestavil teoretickou část (například algoritmus pro konstrukci, klasifikaci a enumeraci uspořádaných pologrup), Petr Gajdoš pak programoval a věnoval se realizaci výpočtů. Podle názoru Petra Gajdoše byl přínos autora nadpoloviční. Kvantitativně tedy autorův přínos k článkům [3, 4] byl 100%, k článku [1] byl 50% a k článku [2] byl 70%.

Poděkování

Děkuji docentu Liboru Polákovi za to, že mne uvedl do světa pologrup, a za všechno, co jsem se naučil pod jeho vedením.

Děkuji Petru Gajdošovi, na nějž jsem se vždy v případě potřeby mohl obrátit jako na výborného programátora.

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1 Uspořádané pologrupy

1.1 Definice

Uspořádaná pologrupa je struktura $(S, \cdot, \leq)$, kde

1. $(S, \cdot)$ je pologrupa
2. $(S, \leq)$ je uspořádaná množina
3. Pro všechna $a, b, c \in S$ platí: Jestliže $a \leq b$ pak $ca \leq cb$ a $ac \leq bc$.

Uveďme pár jednoduchých (a také důležitých) příkladů uspořádaných pologrup. V celé práci budeme symbolem $\mathsf{P}(M)$ označovat potenční množinu množiny M, symbolem $\mathsf{P}_f(M)$ množinu všech konečných neprázdných podmnožin množiny M a symbolem $\mathsf{P}^0_f(M)$ množinu všech konečných podmnožin množiny M.

1. Relaci dělitelnosti na množině přirozených čísel označme $|$ (tedy $a \mid b$ právě když existuje přirozené číslo c splňující $ac = b$). Pak $(\mathbb{N}, \cdot, |)$ je uspořádaný monoid.
2. ([20], XIV. 1., příklad 2) Nechť S je pologrupa. Definujme operaci $\cdot$ na potenční množině $\mathsf{P}(S)$ takto:
$$A \cdot B = AB = \{ab| \; a \in A, b \in B\}.$$
Pak $(\mathsf{P}(S), \cdot, \subseteq)$ je uspořádaná pologrupa.

Základní informace o uspořádaných pologrupách lze najít například v knize [23].

Uvažme jazyk L, který obsahuje jeden operační symbol $\cdot$ četnosti 2 a jeden relační symbol $\leq$ četnosti 2. Jsou tedy uspořádané pologrupy strukturami jazyka L.

V jazyku L uvažme teorii $T = \{x \cdot (y \cdot z) = (x \cdot y) \cdot z,\ x \leq x,\ (x \leq y \;\&\; y \leq x) \to x = y,\ (x \leq y \;\&\; y \leq z) \to x \leq z,\ x \leq y \to z \cdot x \leq z \cdot y,\ x \leq y \to x \cdot z \leq y \cdot z\}$. Struktura A jazyka L je uspořádaná pologrupa právě tehdy, když A je model teorie T.

Třída všech uspořádaných pologrup je kvazivarieta, protože každý axiom teorie T je kvaziidentita (viz například [39], str. 126).

Polosvazově uspořádaná pologrupa je struktura $(S, \cdot, \vee)$, kde

1. $(S, \cdot)$ je pologrupa
2. $(S, \vee)$ je polosvaz
3. Pro všechna $a, b, c \in S$ platí: $c(a \vee b) = ca \vee cb$, $(a \vee b)c = ac \vee bc$.

Ještě obecnější strukturou jsou multiplikativní polosvazy. Jejich definici dostaneme tak, že podmínku 1 v definici polosvazově uspořádané pologrupy nahradíme požadavkem, aby $(S, \cdot)$ byl grupoid. Jak uvádí Birkhoff ([20], XIV.4.), poprvé tento pojem zavedli Dubreil-Jacotin, Lesieur a Croisot v knize [32].

Každý distributivní svaz $(D, \wedge, \vee)$ je polosvazově uspořádaná pologrupa, pokud za operaci $\cdot$ vezmeme operaci $\wedge$.

Na polosvazově uspořádané pologrupy lze nahlížet (a mnozí autoři tak činí) také jako na polookruhy s komutativním a idempotentním sčítáním.

Polosvazově uspořádané pologrupy jsou studovány také v teoretické informatice. O tom se lze dočíst v knize [40] v kapitole "Semirings with particular properties"(kapitola 5, strany 68 až 88).

Třída všech polosvazově uspořádaných pologrup je varieta.

Nechť $(S, \cdot, \vee)$ je polosvazově uspořádaná pologrupa. Na množině S definujme relaci $\leq$ takto: $a \leq b \Leftrightarrow a \vee b = b$. Pak $(S, \cdot, \leq)$ je uspořádaná pologrupa.

Nyní uvedeme tři jednoduché, přirozené a také důležité příklady polosvazově uspořádaných pologrup.

1. ([20], XIV.4., příklad 4) Nechť $(A, \vee)$ je polosvaz. Na množině $\mathsf{End}(A, \vee)$ všech endomorfismů polosvazu $(A, \vee)$ definujme operace $\circ$ a $\vee$ takto:

$$(\alpha \circ \beta)(a) = \alpha(\beta(a)),\ (\alpha \vee \beta)(a) = \alpha(a) \vee \beta(a)$$

   $(\alpha, \beta \in \mathsf{End}(A, \vee),\ a \in A)$. Pak $(\mathsf{End}(A, \vee), \circ, \vee)$ je polosvazově uspořádaná pologrupa.

2. Nechť S je pologrupa. Pak $(\mathsf{P}(S), \cdot, \cup)$ a $(\mathsf{P}_f(S), \cdot, \cup)$ jsou polosvazově uspořádané pologrupy.

3. Nechť A je množina. Pak množina $\mathsf{Rel}(A)$ všech binárních relací na množině A s operacemi skládání a sjednocení je polosvazově uspořádaná pologrupa.

Důležitost uvedených příkladů vyplývá z následujících tří vět.

Věta 1 ([1], Theorem 2.4). *Každá polosvazově uspořádaná pologrupa je izomorfní podstruktuře polosvazově uspořádané pologrupy* $(\mathsf{End}(A, \vee), \circ, \vee)$ *pro vhodný polosvaz* $(A, \vee)$.

Symbolem X budeme značit množinu $X = \{x_1, x_2, x_3, \dots\}$. Volnou pologrupu na množině X označíme X^+.

Věta 2 ([1], Theorem 2.5). $(\mathsf{P}_f(X^+), \cdot, \cup)$ *spolu s vnořením* $\kappa : x \mapsto \{x\}$, $x \in X$, *je volná algebra na množině* X *ve varietě všech polosvazově uspořádaných pologrup.*

Strukturu $(\mathsf{P}_f(X^+), \cdot, \cup)$ budeme dále stručně označovat $X^\square$.

Říkáme, že polosvazově uspořádaná pologrupa $(S, \cdot, \vee)$ je reprezentovatelná binárními relacemi, existuje-li množina A a injektivní homomorfismus $f : (S, \cdot, \vee) \to (\mathsf{Rel}(A), \circ, \cup)$.

Připomeňme, že polosvaz $(A, \vee)$ se nazývá distributivní, pokud pro všechna $x, y, z \in A$, $x \leq y \vee z \Rightarrow x = y' \vee z'$ pro nějaká $y', z' \in A$, $y' \leq y$, $z' \leq z$.

n	1	2	3	4	5	6	7	8	9
$S(n)$	1	5	24	188	1915	28634	1627672	3684030417	105978177936292

Tabulka 1: Počet všech neizomorfních pologrup řádu n

Věta 3 (H. Andréka [18], Theorem 1). *Nechť $(A, \cdot, \vee)$ je polosvazově uspořádaná pologrupa taková, že $(A, \vee)$ je distributivní polosvaz. Pak $(A, \cdot, \vee)$ je reprezentovatelná binárními relacemi.*

V článku [1] je podán přímý a elementární důkaz následujícího důsledku Věty 3.

Důsledek 4 ([1], Theorem 2.7). *Nechť S je pologrupa. Pak polosvazově uspořádaná pologrupa $(\mathsf{P}(S), \cdot, \cup)$ je reprezentovatelná binárními relacemi.*

1.2 Enumerace

V této části budeme především prezentovat výsledky článku [2].

V předchozí části jsme definovali uspořádanou pologrupu. Naskýtá se přirozená otázka: Kolik uspořádaných pologrup existuje v závislosti na počtu prvků (čili řádu)? Přitom izomorfní uspořádané pologrupy budeme při tomto počítání ztotožňovat.

Uspořádané pologrupy $(S_1, \cdot_1, \leq_1)$ a $(S_2, \cdot_2, \leq_2)$ jsou izomorfní, pokud existuje bijekce $\varphi : S_1 \to S_2$ taková, že

1. $\varphi(x \cdot_1 y) = \varphi(x) \cdot_2 \varphi(y)$
2. $x \leq_1 y \iff \varphi(x) \leq_2 \varphi(y)$

pro všechna $x, y \in S_1$. Bijekce φ se nazývá izomorfismus.

Fakt, že struktury A, B téhož jazyka jsou izomorfní, označíme $A \cong B$. Jestliže $(S_1, \cdot_1, \leq_1) \cong (S_2, \cdot_2, \leq_2)$ pak $(S_1, \cdot_1) \cong (S_2, \cdot_2)$ a také $(S_1, \leq_1) \cong (S_2, \leq_2)$. Opačná implikace obecně neplatí.

Počet všech neizomorfních pologrup řádu n označme $S(n)$, počet všech neizomorfních uspořádaných množin řádu n označme $OSet(n)$.

Enumerací pologrup se zabývala řada autorů. Patrně nejvýznamnější práce napsali Plemmons, Jürgensen, Wick, Satoh, Yama, Tokizawa, Distler. Nyní jsou známy hodnoty $S(n)$ pro $n \leq 9$. Hodnotu $S(9)$ poprvé zveřejnil Distler v [27] (viz také [29]). V práci [27] je také uvedena přehledná tabulka obsahující hodnoty $S(n)$ pro $n \leq 9$ (strana 139, Table A.16). Tyto hodnoty lze najít také v [59], posloupnost A027851. My je uvádíme v tabulce 1.

Připomeňme, že pologrupy $(S_1, \cdot_1)$ a $(S_2, \cdot_2)$ se nazývají antiizomorfní, pokud existuje bijekce $\psi : S_1 \to S_2$ taková, že $\psi(x \cdot_1 y) = \psi(y) \cdot_2 \psi(x)$ pro všechna $x, y \in S_1$. Říkáme, že dvě pologrupy jsou ekvivalentní, jestliže jsou izomorfní nebo antiizomorfní.

Hodnota $S(10)$ dosud známa není. Avšak Distler, Jefferson, Kelsey a Kotthoff v [28] již uvádějí, že počet všech neekvivalentních pologrup řádu 10 je 12418001077381302684.

n	$OSet(n)$
1	1
2	2
3	5
4	16
5	63
6	318
7	2045
8	16999
9	183231
10	2567284
11	46749427
12	1104891746
13	33823827452
14	1338193159771
15	68275077901156
16	4483130665195087

Tabulka 2: Počet všech neizomorfních uspořádaných množin řádu n

Enumerací uspořádaných množin se také zabývala řada autorů. Nejdále prozatím postoupili Brinkmann a McKay. V [24] popisují efektivní metodu konstrukce vzájemně neizomorfních uspořádaných množin a také podávají výsledky získané počítačovým programem. Určili $OSet(n)$ pro $n \leq 16$ (viz také [59], posloupnost A000112). Tyto hodnoty jsou zde uvedeny v tabulce 2.

Počet všech neizomorfních uspořádaných pologrup řádu n označme $OS(n)$, počet všech neizomorfních lineárně uspořádaných pologrup řádu n označme $LOS(n)$.

Dříve se autoři věnovali enumeraci uspořádaných pologrup pouze ve dvou extrémních případech, a to pro diskrétně uspořádané pologrupy (tj. pro pologrupy uspořádané relací $=$) a pro lineárně uspořádané pologrupy. Uvědomme si, že uspořádané pologrupy $(S_1, \cdot_1, =)$ a $(S_2, \cdot_2, =)$ jsou izomorfní právě tehdy, když jsou izomorfní pologrupy $(S_1, \cdot_1)$ a $(S_2, \cdot_2)$. To znamená, že tabulka 1 je také tabulkou počtu všech neizomorfních diskrétně uspořádaných pologrup řádu $n \leq 9$.

Jen několik autorů se věnovalo enumeraci lineárně uspořádaných pologrup (Gabovich, Jürgensen, Slaney). Před publikací článku [2] byly známy hodnoty $LOS(n)$ pro $n \leq 8$ ([56], 4.1.5, strana 61, a [59], posloupnost A084965). V [2] jsou nově určeny hodnoty $LOS(9)$ a $LOS(10)$.

My (Petr Gajdoš a autor habilitační práce) se v [2] věnujeme konstrukci, klasifikaci a enumeraci pologrup uspořádaných libovolně.

Potřebujeme vhodnou reprezentaci uspořádaných množin. Pro kladné celé číslo n označíme množinu $\{1, 2, \ldots, n\}$ stručně jako $[n]$. Nechť $(E, \preceq)$ je uspořádaná množina s n prvky. Vezměme bijekci $f : [n] \to E$ takovou, že $f(i) \preceq f(j)$ implikuje $i \leq j$, pro všechna $i, j \in [n]$. Taková bijekce existuje, protože $\preceq$ má lineární rozšíření. Definujeme binární matici $A = (a_{ij})$ typu $n \times n$ takto:

$$a_{ij} = \begin{cases} 1 & f(i) \preceq f(j) \\ 0 & f(i) \not\preceq f(j) \end{cases}$$

pro libovolná $i, j \in [n]$.

Známe-li matici A a bijekci f, pak známe relaci $\preceq$. Jestliže ztotožníme $f(i)$ a i (tj. prvek $f(i)$ označíme jako i), pak $E = [n]$ a $i \preceq j$ implikuje $i \leq j$ a

$$a_{ij} = \begin{cases} 1 & i \preceq j \\ 0 & i \not\preceq j \end{cases}$$

pro libovolná $i, j \in [n]$.

Všimněme si, že všechny prvky matice A ležící pod hlavní diagonálou jsou rovny 0 a všechny prvky matice A ležící na hlavní diagonále jsou rovny 1. V důsledku toho kompletní informace o relaci $\preceq$ je obsažena v binární posloupnosti délky $\frac{n^2-n}{2}$, totiž v posloupnosti

$$a_{12} \dots a_{1n} \cdot a_{23} \dots a_{2n} \dots a_{n-2,n-1} a_{n-2,n} \cdot a_{n-1,n}.$$

Tato notace pomocí binární posloupnosti slouží hlavně k zápisu uspořádané množiny v sevřeném tvaru, což je vhodné pro tabulku 3 v [2] (strany 649 – 659). Aby se tyto posloupnosti lépe četly, klademe tečku mezi a_{in} a $a_{i+1,i+2}$ (pro $1 \leq i \leq n-2$).

Množinu všech automorfismů uspořádané množiny E označíme $\mathsf{Aut}(E)$. Množina $\mathsf{Aut}(E)$ je podgrupou symetrické grupy $\mathsf{Sym}(E)$ množiny E. Jestliže E je složena z prvků $1, 2, \dots, n$, pak $\mathsf{Aut}(E)$ je podgrupa symetrické grupy $\mathsf{Sym}(n)$.

Důležitost grup automorfismů uspořádaných množin pro naše účely ukazuje třeba tento fakt: Nechť $(S, \preceq)$ je uspořádaná množina. Pak uspořádané pologrupy $(S, *, \preceq)$, $(S, \circ, \preceq)$ jsou izomorfní právě tehdy, když existuje $\pi \in \mathsf{Aut}((S, \preceq))$ tak, že $\pi(x * y) = \pi(x) \circ \pi(y)$ (pro všechna $x, y \in S$).

Pro kladné celé číslo n uvažme rozklad množiny všech uspořádaných množin s prvky $1, \dots, n$ podle ekvivalence $\cong$. Množinu reprezentantů všech tříd ekvivalence označíme $\mathcal{OSET}(n)$.

V článku [2] jsme popsali konstrukci všech neizomorfních uspořádaných pologrup řádu nejvýše 7. Pro $n \in \{1, 2, 3, 4, 5, 6, 7\}$ jsme potřebovali znát $\mathcal{OSET}(n)$ a grupy automorfismů všech prvků z $\mathcal{OSET}(n)$.

Nechť A je binární matice typu $n \times n$. Definujeme binární relaci ρ_A na množině $[n]$ následovně: $i \rho_A j$ právě tehdy, když $a_{ij} = 1$ (pro $i, j \in [n]$).

Množinu $\mathcal{OSET}(n)$ jsme určili takto:

S využitím počítače jsme našli binární matice $A_1, A_2, \dots, A_l$ typu $n \times n$ s 0 pod hlavní diagonálou a s 1 na hlavní diagonále takové že

$$([n], \rho_{A_1}), ([n], \rho_{A_2}), \dots, ([n], \rho_{A_l})$$

jsou všechny neizomorfní uspořádané množiny s n prvky. Máme $\mathcal{OSET}(n) = \{A_1, A_2, \ldots, A_l\}$. Samozřejmě, $l = OSet(n)$.

Naše konstrukce množiny $\mathcal{OSET}(n)$ nebyla efektivní, byla provedena metodou hrubé síly. Efektivní metodu konstrukce vzájemně neizomorfních uspořádaných množin popsali Brinkmann a McKay v [24].

Dále jsme pro všechna $A \in \mathcal{OSET}(n)$ $(n \in \{1, 2, 3, 4, 5, 6, 7\})$ určili grupu $\mathsf{Aut}(A)$. Pro každou permutaci $\pi \in \mathsf{Sym}(n)$ jsme testovali, zda $i\rho_A j$ a $\pi(i)\rho_A\pi(j)$ jsou ekvivalentní pro všechna $i, j \in [n]$. Permutace splňující uvedenou podmínku patří do $\mathsf{Aut}(A)$.

Uvedená metoda opět není efektivní, je to metoda hrubé síly. Pfeiffer v [42] určil počet všech neizomorfních uspořádaných množin s n prvky pro $n \leq 12$ a také určil jejich grupy automorfismů (viz také [59], posloupnost A091070).

Nechť n je kladné celé číslo. Všechny neizomorfní uspořádané pologrupy řádu n jsme sestrojili ve dvou krocích: Nejprve jsme určili množinu

$$\mathcal{OSET}(n) = \{([n], \preceq_1), ([n], \preceq_2), \ldots, ([n], \preceq_{OSet(n)})\}.$$

Konstrukce množiny $\mathcal{OSET}(n)$ je popsána výše. Pak jsme pro každé $i \in \{1, \ldots, OSet(n)\}$ našli všechny neizomorfní uspořádané pologrupy tvaru $([n], *, \preceq_i)$. Označíme je $([n], *_{i1}, \preceq_i), \ldots, ([n], *_{ip_i}, \preceq_i)$. Pak

$$([n], *_{ij}, \preceq_i) \ i = 1, \ldots, OSet(n), \ j = 1, \ldots, p_i$$

jsou všechny neizomorfní uspořádané pologrupy s n prvky.

Nechť S je pologrupa. Pro kladné celé číslo n definujeme množiny S^n rekurzivně následujícím způsobem:

1. $S^1 = S$
2. $S^n = SS^{n-1}$ pro $n > 1$.

Nechť k je kladné celé číslo. Připomeňme, že k-nilpotentní pologrupa je pologrupa S s vlastnostmi $card(S^k) = 1$ a $card(S^l) > 1$ pro všechna celá čísla l splňující $0 < l < k$. Tudíž 1-nilpotentní pologrupy jsou přesně pologrupy s jedním prvkem a pologrupa S je 2-nilpotentní právě tehdy, když $card(S) > 1$ a existuje $b \in S$ takové, že $xy = b$ pro všechna $x, y \in S$.

Ještě než zformulujeme hlavní algoritmus z článku [2], uvedeme formuli pro $O2NS(n)$, kde $O2NS(n)$ značí počet všech neizomorfních uspořádaných 2-nilpotentních pologrup řádu n.

Tvrzení 5 ([2], strana 644). *Nechť n je kladné celé číslo. Pak*

$$O2NS(n) = \sum_{E \in \mathcal{OSET}(n)} \left(\frac{1}{card(\mathsf{Aut}(E))} \cdot \sum_{\tau \in \mathsf{Aut}(E)} F(\tau) \right),$$

n	$O2NS(n)$
2	3
3	11
4	47
5	243
6	1533
7	12038
8	118818
9	1487301
10	23738557
11	484673601
12	12677658783

Tabulka 3: Počet všech neizomorfních uspořádaných 2-nilpotentních pologrup řádu n

kde $F(\tau)$ je počet pevných bodů permutace τ.

Použitím formule z Tvrzení 5 byly vypočteny hodnoty $O2NS(n)$ pro $n \leq 12$. Zde je uvádíme v tabulce 3. Byli jsme schopni vypočítat hodnoty $O2NS(n)$ pro $n \leq 7$ a hodnoty $O2NS(n)$ pro $n \leq 12$ na naši žádost laskavě vypočítal Goetz Pfeiffer ze svých údajů o grupách automorfismů uspořádaných množin.

Uvedené hodnoty ukazují rozdíl mezi pologrupami a uspořádanými pologrupami. Například, existuje pouze jedna 2-nilpotentní pologrupa se 7 prvky, ale existuje 12038 uspořádaných 2-nilpotentních pologrup se 7 prvky.

Nyní již konečně uvedeme hlavní algoritmus článku [2] a poté budeme komentovat výsledky výpočtů.

Nechť n je kladné celé číslo. Nechť $([n], \preceq)$ je uspořádaná množina. Chceme sestrojit všechny neizomorfní uspořádané pologrupy tvaru $([n], *, \preceq)$.

Nechť $M_n(U)$ značí množinu všech matic typu $n \times n$ nad množinou U. Každá matice $A = (a_{ij}) \in M_n([n])$ určuje binární operaci $*_A$ na $[n]$ předpisem

$$i *_A j = a_{ij}$$

$(i, j \in [n])$. Takže A je Cayleyho tabulka operace $*_A$.

Pro každou permutaci $\pi \in \mathsf{Sym}(n)$ a každou matici $A \in M_n([n])$ definujeme

$$\pi(A) = B = (b_{ij}) \in M_n([n]),$$

kde

$$b_{ij} = \pi(a_{\pi^{-1}(i),\pi^{-1}(j)})$$

(pro $i, j \in [n]$).

Označme $C_n([n], \preceq)$ množinu všech $A \in M_n([n])$ takových, že operace $*_A$ je asociativní a kompatibilní s relací $\preceq$ (tj. pro všechna $i, j, k \in [n]$, $i \preceq j$ implikuje $i *_A k \preceq j *_A k$ a $k *_A i \preceq k *_A j$). Takže

$$C_n([n], \preceq) = \{A \in M_n([n]) |\ ([n], *_A, \preceq) \text{ je usp. pologrupa}\}.$$

Definujeme binární relaci $\equiv$ na množině $C_n([n], \preceq)$ následovně: Pro $A, B \in C_n([n], \preceq)$, $A \equiv B$ tehdy a jen tehdy, když $\pi(A) = B$ pro nějaké $\pi \in \mathsf{Aut}(([n], \preceq))$. Platí

Lemma 6 ([2], Lemma 3.4). *Relace $\equiv$ je relace ekvivalence na množině $C_n([n], \preceq)$ a pro všechna $A, B \in C_n([n], \preceq)$ platí: $([n], *_A, \preceq) \cong ([n], *_B, \preceq)$ právě tehdy, když $A \equiv B$.*

Zajímá nás tedy faktorová množina $C_n([n], \preceq)/\equiv$.

Nechť $C_n([n], \preceq)/\equiv\ = \{\mathcal{T}_1, \mathcal{T}_2 \ldots, \mathcal{T}_p\}$. Uvědomme si, že p je počet všech neizomorfních uspořádaných pologrup tvaru $([n], *, \preceq)$.

Množina $M_n([n])$ je lineárně uspořádána relací $\leq$, která je definována následujícím pravidlem: Nechť $A, B \in M_n([n])$. Pak

$$A \leq B$$

právě když $a_{ij} = b_{ij}$ pro všechna $i, j \in [n]$ nebo existuje dvojice $k, l \in [n]$ tak, že

$$a_{kl} < b_{kl}$$

(symbol $<$ zde označuje ostré obvyklé uspořádání celých čísel) a $a_{ij} = b_{ij}$ pro všechna $i, j \in [n]$ splňující

$$j + (i - 1) \cdot n < l + (k - 1) \cdot n.$$

Jako obvykle, $A < B$ znamená $A \leq B$ a $A \neq B$.

Sestrojili jsme matice $A_1, A_2, \ldots, A_p$, které pro každé $i \in \{1, 2, \ldots, p\}$ splňují

1. $A_i \in \mathcal{T}_i$,

2. $A_i \leq B$ pro všechna $B \in \mathcal{T}_i$.

Nechť $A = M_n([n] \cup \{0\})$. Na matici A se můžeme dívat jako na tabulku parciální binární operace na množině $[n]$. Nechť $i, j \in [n]$. Jestliže $a_{ij} = 0$, pak prvek $i *_A j$ není definován. Jestliže $a_{ij} \neq 0$, pak $i *_A j = a_{ij}$.

Nechť $A \in M_n([n] \cup \{0\})$. Definujeme množinu matic $R(A)$ předpisem

$$R(A) = \{B \in M_n([n]) |\ \text{pro lib. } i, j \in [n], a_{ij} \neq 0 \text{ implikuje } b_{ij} = a_{ij}\}.$$

Tudíž $R(A)$ je množina všech matic, které získáme tak, že všechny nuly v A nahradíme prvky množiny $[n]$.

Nechť opět $A = (a_{ij}) \in M_n([n] \cup \{0\})$. Nechť $k \in [n]$. Definujeme matici $k(A) = B = (b_{ij}) \in M_n([n])$ předpisem

$$b_{ij} = \begin{cases} a_{ij} & a_{ij} \neq 0 \\ k & a_{ij} = 0 \end{cases}$$

Tudíž $k(A)$ je matice, kterou získáme tak, že všechny nuly v A nahradíme prvkem k. Zřejmě $k(A) \in R(A)$. Pokud $A \in M_n([n])$, je $k(A) = A$.

Je zřejmé, že

$$1(A) \leq B \leq n(A)$$

pro všechna $B \in R(A)$.

Pro každou permutaci $\pi \in \mathsf{Sym}(n)$ a každou matici $A \in M_n([n] \cup \{0\})$ definujeme

$$\overline{\pi}(A) = B = (b_{ij}) \in M_n([n] \cup \{0\}),$$

kde

$$b_{ij} = \begin{cases} \pi(a_{\pi^{-1}(i),\pi^{-1}(j)}) & a_{\pi^{-1}(i),\pi^{-1}(j)} \neq 0 \\ 0 & a_{\pi^{-1}(i),\pi^{-1}(j)} = 0 \end{cases}$$

Jestliže $A \in M_n([n])$, pak $\overline{\pi}(A) = \pi(A)$.

Náš algoritmus je založen na myšlenkách, které prezentoval Plemmons v [47]. Rozdíl spočívá v tom, že my používáme grupu automorfismů $\mathsf{Aut}(([n], \preceq))$ místo symetrické grupy $\mathsf{Sym}(n)$.

Algoritmus 7 ([2], Algorithm 3.5). Vstup: Uspořádaná množina $([n], \preceq)$ pro nějaké kladné celé číslo n.

Výstup: matice $A_1, \ldots, A_p \in M_n([n])$ takové, že $([n], *_{A_1}, \preceq), \ldots, ([n], *_{A_p}, \preceq)$ jsou všechny neizomorfní uspořádamé pologrupy tvaru $([n], *, \preceq)$.

Algoritmus pracuje s maticí $A = (a_{kl}) \in M_n([n] \cup \{0\})$ a s celými čísly $q, i, j, 1 \leq i \leq n, 1 \leq j \leq n$.

1. (Inicializace.) $a_{kl} \leftarrow 1$ pro všechna $k, l \in [n]$, $q \leftarrow 1$, $i \leftarrow n$, $j \leftarrow n$, $A_q \leftarrow A$.
2. Jestliže $a_{ij} < n$, pak $a_{ij} \leftarrow a_{ij} + 1$ a jdi na krok 4.
3. (Máme $a_{ij} = n$.)
   (i) $a_{ij} \leftarrow 0$
   (ii) Jestliže $j > 1$, pak $j \leftarrow j - 1$ a jdi na krok 2.
   (iii) (Máme $j = 1$.) Jestliže $i > 1$, pak $i \leftarrow i - 1$, $j \leftarrow n$ a jdi na krok 2.
   (iv) (Máme $j = i = 1$.) Výpočet končí a máme všechny matice $A_1, A_2, \ldots, A_p$.
4. (Test asociativity.) Jestliže $a_{lm} \neq 0$, $a_{k,a_{lm}} \neq 0$, $a_{kl} \neq 0$, $a_{a_{kl},m} \neq 0$ a $a_{k,a_{lm}} \neq a_{a_{kl},m}$ pro nějaká $k, l, m \in [n]$, pak jdi na krok 2.

5. (Test kompatibility.) Jestliže $k \preceq l$, $a_{mk} \neq 0$, $a_{ml} \neq 0$, $a_{mk} \not\preceq a_{ml}$ nebo $k \preceq l$, $a_{km} \neq 0$, $a_{lm} \neq 0$, $a_{km} \not\preceq a_{lm}$ pro nějaká $k, l, m \in [n]$, pak jdi na krok 2.

6. (Test izomorfismu.) Jestliže $n(\overline{\pi}(A)) < 1(A)$ pro nějaké $\pi \in \mathsf{Aut}(([n], \preceq))$, pak jdi na krok 2.

7. (i) Jestliže $j < n$, pak $j \leftarrow j + 1$ a jdi na krok 2.
   (ii) (Máme $j = n$.) Jestliže $i < n$, pak $i \leftarrow i + 1$, $j \leftarrow 1$ a jdi na krok 2.
   (iii) (Máme $j = i = n$.) $q \leftarrow q + 1$, $A_q \leftarrow A$ a jdi na krok 2.

Abychom sestrojili všechny neizomorfní uspořádané pologrupy s n prvky, musíme najít všechny neizomorfní uspořádané množiny $([n], \preceq_i)$ s n prvky, $i = 1, \dots, OSet(n)$, a pak aplikovat Algoritmus 7 na každou uspořádanou množinu $([n], \preceq_i)$, $i = 1, \dots, OSet(n)$. Získané uspořádané pologrupy jsou roztříděny podle typu svého uspořádání.

Popsaným způsobem se nám podařilo sestrojit všechny neizomorfní uspořádané pologrupy s n prvky, kde $n \leq 7$.

V [2], Table 3, jsou uvedeny počty uspořádaných pologrup s n prvky postupně pro $n = 1, 2, 3, 4, 5, 6$. Binární posloupnost v prvním sloupci určuje relaci uspořádání na $[n]$ a to způsobem, který jsme již výše popsali. Symboly **s**, **c**, **m**, **b**, **r**, **i**, **2**, **3** v záhlaví tabulky jsou zkratky:

- **s** – pologrupa
- **c** – komutativní pologrupa ($xy = yx$ pro všechna $x, y \in S$)
- **m** – monoid (S má neutrální prvek)
- **b** – band ($x^2 = x$ pro všechna $x \in S$)
- **r** – regulární pologrupa (pro všechna $x \in S$ existuje prvek $y \in S$ takový, že $xyx = x$ a $yxy = y$)
- **i** – inverzní pologrupa (pro všechna $x \in S$ existuje právě jeden prvek $y \in S$ takový, že $xyx = x$ a $yxy = y$)
- **2** – 2-nilpotentní pologrupa ($card(S^2) = 1$ a $card(S) > 1$)
- **3** – 3-nilpotentní pologrupa ($card(S^3) = 1$ a $card(S^2) > 1$)

Jako příklad uveďme jeden řádek z tabulky Table 3:

	s	**c**	**m**	**b**	**r**	**i**	**2**	**3**
1111.000.00.0	1740	277	199	244	313	44	2	118

počet prvků	1	2	3	4	5	6	7
pologrupy	1	11	173	4753	198838	13457454	4207546916
komutativní pologrupy	1	7	83	1468	37248	1337698	71748346
monoidy	1	4	37	549	13371	504634	32113642
bandy	1	7	69	1035	20305	494848	14349957
regulární pologrupy	1	8	76	1149	22419	546386	15842224
inverzní pologrupy	1	4	26	239	2886	44275	830584
2-nilpotentní pologrupy	0	3	11	47	243	1533	12038
3-nilpotentní pologrupy	0	0	7	246	14150	2561653	3215028097

Tabulka 4: Celkové počty neizomorfních uspořádaných pologrup

Uvedený řádek říká, že existuje 1740 neizomorfních uspořádaných pologrup obsahujících nejmenší prvek a další čtyři prvky, které jsou vzájemně neporovnatelné. Dále existuje celkem 277 neizomorfních uspořádaných komutativních pologrup obsahujících nejmenší prvek a další čtyři prvky, které jsou vzájemně neporovnatelné, atd.

V Table 3 není prezentována enumerace uspořádaných pologrup řádu 7, protože by to k tabulce přidalo dalších $OSet(7) = 2045$ řádků. Najdeme však v [2] hodnoty, jež zde uvádíme v tabulce 4.

V [2] je také algoritmus (Algorithm 3.6), v němž na vstupu je uspořádaná množina $([n], \leq)$, kde $1 < 2 < \cdots < n$ pro nějaké kladné celé číslo n, a na výstupu jsou matice $A_1, \ldots, A_p \in M_n([n])$ takové, že $([n], *_{A_1}, \leq), \ldots, ([n], *_{A_p}, \leq)$ jsou všechny neizomorfní uspořádané pologrupy tvaru $([n], *, \leq)$. Tento algoritmus je modifikací Algoritmu 7 a podařilo se nám pomocí něj sestrojit všechny neizomorfní lineárně uspořádané pologrupy s n prvky, kde $n \leq 10$. Celkové počty lineárně uspořádaných pologrup jsou uvedeny v [2], Table 5, a to v členění obdobném členění tabulky 4.

Nechť n je kladné celé číslo. Připomeňme, že počet všech neizomorfních pologrup s n prvky jsme označili $S(n)$ a počet všech neizomorfních uspořádaných pologrup s n prvky jsme označili $OS(n)$. Počet všech neizomorfních lineárně uspořádaných pologrup s n prvky označíme $LOS(n)$. Hodnoty $S(n)$, $LOS(n)$ a $OS(n)$ porovnáváme v tabulce 5.

Hodnoty $S(n)$ jsou známy pro $n \leq 9$ ([27], Table A.16). Hodnoty $LOS(n)$ byly známy pro $n \leq 8$ ([56], 4.1.5, strana 61, a [59], posloupnost A084965). My jsme určili hodnoty $LOS(n)$ pro $n \leq 10$ a hodnoty $OS(n)$ pro $n \leq 7$.

2 Variety uspořádaných pologrup

2.1 Úplně invariantní stabilní kvaziuspořádání

V části 2 budeme především prezentovat výsledky článku [3].

n	$S(n)$	$LOS(n)$	$OS(n)$
1	1	1	1
2	5	6	11
3	24	44	173
4	188	386	4753
5	1915	3852	198838
6	28634	42640	13457454
7	1627672	516791	4207546916
8	3684030417	6817378	?
9	105978177936292	98091071	?
10	?	1569786228	?

Tabulka 5: Počty neizomorfních pologrup, lineárně uspořádaných pologrup a uspořádaných pologrup

Připomeneme nejdříve některé základní pojmy a poznatky univerzální algebry.

Nechť je dán jazyk L. Většinou budeme strukturu jazyka L a její nosič označovat stejným symbolem. Není to striktně vzato správné, ale jednak se to tak někdy dělá a jednak se domníváme, že to nepovede k nedorozumění.

Nechť A, B jsou struktury jazyka L.

Zobrazení $f : A \to B$ se nazývá izomorfismus A na B, platí-li:

1. f je bijekce
2. pro každý konstantní symbol c jazyka L je $f(c^A) = c^B$
3. pro každý operační symbol $*$ jazyka L četnosti n (n je kladné celé číslo) a pro všechna $a_1, a_2, \ldots, a_n \in A$ je $f(*^A(a_1, a_2, \ldots, a_n)) = *^B(f(a_1), f(a_2), \ldots, f(a_n))$
4. pro každý relační symbol $\ll$ jazyka L četnosti n a pro všechna $a_1, a_2, \ldots, a_n \in A$ platí: $(a_1, a_2, \ldots, a_n) \in\ll^A$ právě tehdy, když $(f(a_1), f(a_2), \ldots, f(a_n)) \in\ll^B$.

Říkáme, že struktury A, B jsou izomorfní (zápis: $A \cong B$), jestliže existuje nějaký izomorfismus A na B.

Zobrazení $f : A \to B$ se nazývá homomorfismus A do B, platí-li:

1. pro každý konstantní symbol c jazyka L je $f(c^A) = c^B$
2. pro každý operační symbol $*$ jazyka L četnosti n (n je kladné celé číslo) a pro všechna $a_1, a_2, \ldots, a_n \in A$ je $f(*^A(a_1, a_2, \ldots, a_n)) = *^B(f(a_1), f(a_2), \ldots, f(a_n))$
3. pro každý relační symbol $\ll$ jazyka L četnosti n a pro všechna $a_1, a_2, \ldots, a_n \in A$ platí: $(a_1, a_2, \ldots, a_n) \in\ll^A$ implikuje $(f(a_1), f(a_2), \ldots, f(a_n)) \in\ll^B$.

Endomorfismem struktury A rozumíme homomorfismus A do A. Množinu všech endomorfismů struktury A budeme značit $\mathsf{End}(A)$.

Říkáme, že struktura B je homomorfním obrazem struktury A, jestliže existuje homomorfismus A na B.

Říkáme, že struktura B je podstrukturou struktury A, platí-li:

1. $B \subseteq A$
2. pro každý konstantní symbol c jazyka L je $c^B = c^A$
3. pro každý operační symbol $*$ jazyka L četnosti n (n je kladné celé číslo) a pro všechna $b_1, b_2, \ldots, b_n \in B$ je $*^B(b_1, b_2, \ldots, b_n) = *^A(b_1, b_2, \ldots, b_n)$
4. pro každý relační symbol $\ll$ jazyka L četnosti n a pro všechna $b_1, b_2, \ldots, b_n \in B$ platí: $(b_1, b_2, \ldots, b_n) \in \ll^B$ právě tehdy, když $(b_1, b_2, \ldots, b_n) \in \ll^A$.

Nechť A_i, $i \in I$, je množina struktur jazyka L. Kartézským (přímým) součinem struktur A_i, $i \in I$, rozumíme strukturu A jazyka L s nosičem $\prod_{i \in I} A_i$, přičemž platí:

1. pro každý konstantní symbol c jazyka L a pro každé $i \in I$ je $\pi_i(c^A) = c^{A_i}$
2. pro každý operační symbol $*$ jazyka L četnosti n (n je kladné celé číslo), pro všechna $a_1, a_2, \ldots, a_n \in A$ a každé $i \in I$ je $\pi_i(*^A(a_1, a_2, \ldots, a_n)) = *^{A_i}(\pi_i(a_1), \pi_i(a_2), \ldots, \pi_i(a_n))$
3. pro každý relační symbol $\ll$ jazyka L četnosti n a pro všechna $a_1, a_2, \ldots, a_n \in A$ platí: $(a_1, a_2, \ldots, a_n) \in \ll^A$ právě tehdy, když pro každé $i \in I$ je $(\pi_i(a_1), \pi_i(a_2), \ldots, \pi_i(a_n)) \in \ll^{A_i}$.

Zavedeme operátory $\mathsf{I}, \mathsf{H}, \mathsf{S}, \mathsf{Pr}$ zobrazující třídy struktur jazyka L do tříd struktur jazyka L. Nechť $\mathcal{C}$ je třída struktur jazyka L. Nechť A je struktura jazyka L. Pak

- $A \in \mathsf{I}(\mathcal{C})$ právě tehdy, když A je izomorfní s nějakou strukturou ze třídy $\mathcal{C}$
- $A \in \mathsf{H}(\mathcal{C})$ právě tehdy, když A je homomorfním obrazem nějaké struktury ze třídy $\mathcal{C}$
- $A \in \mathsf{S}(\mathcal{C})$ právě tehdy, když A je podstruktura nějaké struktury ze třídy $\mathcal{C}$
- $A \in \mathsf{Pr}(\mathcal{C})$ právě tehdy, když A je kartézským součinem nějakého souboru struktur ze třídy $\mathcal{C}$.

Nechť $\mathcal{C}$ je třída struktur jazyka L. Volnou strukturou ve třídě $\mathcal{C}$ na neprázdné množině Z rozumíme uspořádanou dvojici (F, ι), kde $F \in \mathcal{C}$ a $\iota : Z \to F$ je zobrazení s následující univerzální vlastností: pro každou strukturu $A \in \mathcal{C}$ a každé zobrazení $\vartheta : Z \to A$ existuje jediný homomorfismus $\psi : F \to A$ takový, že $\iota\psi = \vartheta$. Nechť (F_1, ι_1), (F_2, ι_2) jsou volné struktury ve třídě $\mathcal{C}$ na množině Z. Pak existuje izomorfismus $\varphi : F_1 \to F_2$ takový, že $\iota_1\varphi = \iota_2$.

V případech, kdy zobrazení ι je zřejmé, budeme většinou ι vynechávat a budeme stručně říkat, že F je volná struktura v $\mathcal{C}$ na Z.

Třída $\mathcal{C}$ struktur jazyka L je netriviální, pokud obsahuje aspoň jednu aspoň dvouprvkovou strukturu.

Nechť $\mathcal{C}$ je netriviální třída struktur jazyka L. Nechť (F, ι) je volná struktura ve třídě $\mathcal{C}$ na neprázdné množině Z. Pak zobrazení ι je prosté. Po ztotožnění z a $\iota(z)$, pro každé $z \in Z$, bude $Z \subseteq F$.

Buď Y neprázdná množina. Symbolem Y^+ budeme značit množinu všech neprázdných slov nad abecedou Y. Prázdné slovo budeme značit ε. Položíme $Y^* = Y^+ \cup \{\varepsilon\}$. Množina Y^+ spolu s operací skládání slov je pologrupa. Množina Y^* spolu s operací skládání slov je monoid (neutrálním prvkem je ε). Třídu všech pologrup budeme značit $\mathbf{S}$, třídu všech monoidů budeme značit $\mathbf{M}$. Pak pologrupa Y^+ je volná v $\mathbf{S}$ na množině Y a monoid Y^* je volný v $\mathbf{M}$ na množině Y.

Následující věta zaručuje, že v jistých třídách struktur jazyka L existují volné struktury na každé neprázdné množině.

Věta 8 (J. Ježek [39], Věta 16.6). *Nechť $\mathcal{C}$ je netriviální třída struktur jazyka L splňující $\mathsf{I}(\mathcal{C}) \subseteq \mathcal{C}$, $\mathsf{S}(\mathcal{C}) \subseteq \mathcal{C}$, $\mathsf{Pr}(\mathcal{C}) \subseteq \mathcal{C}$. Pak pro každou neprázdnou množinu Z existuje v $\mathcal{C}$ volná struktura na množině Z.*

Uvažme nyní jazyk L, který obsahuje jeden funkční symbol $\cdot$ četnosti 2 a jeden relační symbol $\leq$ četnosti 2. Třídu všech uspořádaných pologrup budeme značit $\mathbf{OS}$. Zřejmě $\mathbf{OS}$ je třída struktur jazyka L. Snadno lze ukázat, že $\mathsf{I}(\mathbf{OS}) \subseteq \mathbf{OS}$, $\mathsf{S}(\mathbf{OS}) \subseteq \mathbf{OS}$, $\mathsf{Pr}(\mathbf{OS}) \subseteq \mathbf{OS}$. Na základě Věty 8 pro každou neprázdnou množinu Z existuje volná uspořádaná pologrupa na Z. Tento fakt také plyne z [3], Theorem 2.5. O tom podrobněji pojednáme dále.

Připomeňme, že jazyk neobsahující relační symboly se nazývá typ. Struktury, jejichž jazyk je typem, se nazývají algebry.

Třída $\mathcal{V}$ algeber typu τ se nazývá varieta, platí-li: $\mathsf{H}(\mathcal{V}) \subseteq \mathcal{V}$, $\mathsf{S}(\mathcal{V}) \subseteq \mathcal{V}$ a $\mathsf{Pr}(\mathcal{V}) \subseteq \mathcal{V}$.

Z Věty 8 ihned plyne, že v každé netriviální varietě existuje volná algebra na libovolné neprázdné množině.

Platí následující klasická

Věta 9 (Birkhoffova) (G. Birkhoff [20], VI, Theorem 22; S. Burris, H. P. Sankappanavar [25], Theorem 11.9). *Třída algeber daného typu τ je varietou právě tehdy, když je třídou všech modelů nějaké teorie v jazyku τ, jejíž každý axiom je identitou.*

Připomeňme si, že identita je formule $s = t$, kde s a t jsou libovolné termy téhož jazyka (typu).

Připomeňme ještě, že kongrunce ρ algebry A se nazývá úplně invariantní, pokud $a\rho b$ implikuje $f(a)\rho f(b)$ pro všechna $a, b \in A$ a pro každý endomorfismus f algebry A. Množinu všech úplně invariantních kongruencí algebry A budeme značit $\mathsf{FIC}(A)$.

Již v části 1.1 jsme symbol X rezervovali pro množinu $\{x_1, x_2, x_3, \dots\}$. Tedy X je nekonečná spočetná množina. Prvky množiny X se nazývají proměnné.

Zvolme nyní libovolně nějakou netriviální varietu $\mathcal{V}$ algeber typu τ. Víme, že ve $\mathcal{V}$ existuje volná algebra na množině X. Tuto volnou algebru označme F.

F-identitou (stručně: identitou) budeme rozumět uspořádanou dvojici prvků množiny F. Řekneme, že identita (u, v) platí (je splněna) v algebře A typu τ pokud $f(u) = f(v)$ pro každý homomorfismus $f : F \to A$.

Buď $\Sigma \subseteq F \times F$. Označme $\mathsf{Mod}_{\mathcal{V}}(\Sigma)$ třídu všech algeber z variety $\mathcal{V}$ splňujících každou identitu z množiny Σ. Snadno lze dokázat, že $\mathsf{Mod}_{\mathcal{V}}(\Sigma)$ je varieta.

Buď $\mathcal{C} \subseteq \mathcal{V}$. Označme $\mathsf{Eq}_{\mathcal{V}}(\mathcal{C})$ množinu všech identit platících v každé algebře ze třídy $\mathcal{C}$. Snadno lze dokázat, že $\mathsf{Eq}_{\mathcal{V}}(\mathcal{C})$ je úplně invariantní kongruencí algebry F.

Nyní zformulujeme důležitou větu.

Věta 10 (J. Ježek [39], Věta 39.1; viz také G. M. Bergman [19], Proposition 8.6.4). *Nechť $\mathcal{V}$ je netriviální varieta algeber typu τ. Nechť F je volná algebra ve varietě $\mathcal{V}$ na množině X. Pravidla*

$$\mathcal{W} \mapsto \mathsf{Eq}_{\mathcal{V}}(\mathcal{W}), \ \ \rho \mapsto \mathsf{Mod}_{\mathcal{V}}(\rho)$$

určují vzájemně inverzní bijekce mezi všemi podvarietami variety $\mathcal{V}$ a všemi úplně invariantními kongruencemi algebry F. Přitom

1. *pro libovolné variety $\mathcal{W}_1$, $\mathcal{W}_2$ algeber typu τ, $\mathcal{W}_1 \subseteq \mathcal{V}$, $\mathcal{W}_2 \subseteq \mathcal{V}$, platí:*

$$\mathcal{W}_1 \subseteq \mathcal{W}_2 \Longleftrightarrow \mathsf{Eq}_{\mathcal{V}}(\mathcal{W}_1) \supseteq \mathsf{Eq}_{\mathcal{V}}(\mathcal{W}_2)$$

2. *pro libovolné úplně invariantní kongruence ρ_1, ρ_2 algebry F platí:*

$$\rho_1 \subseteq \rho_2 \Longleftrightarrow \mathsf{Mod}_{\mathcal{V}}(\rho_1) \supseteq \mathsf{Mod}_{\mathcal{V}}(\rho_2).$$

Speciálně tedy máme bijekci mezi všemi podvarietami variety $\mathbf{S}$ a všemi úplně invariantními kongruencemi pologrupy X^+. V této souvislosti místo $\mathsf{Eq}_{\mathbf{S}}(\mathcal{W})$ píšeme často $\rho_{\mathcal{W}}$.

Ukončíme nyní přehled základních pojmů a tvrzení univerzální algebry, jež jsou důležité pro naše účely, a začneme (konečně) prezentovat hlavní výsledky našeho článku [3].

V [3] je formulována a dokázána analogie Věty 10 pro třídu **OS**.

Musíme nejprve vyjasnit, jaké podtřídy třídy **OS** budou hrát roli podvariet.

Buď nyní L jazyk, který má jeden operační symbol $\cdot$ četnosti 2 a jeden relační symbol $\leq$ četnosti 2.

Uspořádané pologrupy jsou strukturami jazyka L. Zřejmě $\mathsf{I}(\mathbf{OS}) \subseteq \mathbf{OS}$, $\mathsf{S}(\mathbf{OS}) \subseteq \mathbf{OS}$, $\mathsf{Pr}(\mathbf{OS}) \subseteq \mathbf{OS}$. Definujme struktury A, B jazyka L takto: A i B mají nosič $\{0, 1\}$ a $\cdot^A = \cdot^B = \cdot$, $\leq^A = \{(0,0), (1,1)\}$, $\leq^B = \{(0,0), (0,1), (1,0), (1,1)\}$. Pak $A \in \mathbf{OS}$. Identické zobrazení je homomorfismus A na B. Takže $B \in \mathsf{H}(\mathbf{OS})$. Ovšem $B \notin \mathbf{OS}$, protože relace $\leq^B$ není antisymetrická. Ukázali jsme, že $\mathsf{H}(\mathbf{OS}) \nsubseteq \mathbf{OS}$. Roli podvariet třídy $\mathbf{OS}$ nebudou tedy hrát podtřídy $\mathcal{C} \subseteq \mathbf{OS}$ splňující $\mathsf{H}(\mathcal{C}) \subseteq \mathcal{C}$, $\mathsf{S}(\mathcal{C}) \subseteq \mathcal{C}$, $\mathsf{Pr}(\mathcal{C}) \subseteq \mathcal{C}$.

Budeme operátor H relativizovat vzhledem ke třídě $\mathbf{OS}$. Buď $\mathcal{C}$ třída struktur jazyka L. Pak klademe $\mathsf{H}_{\mathbf{OS}}(\mathcal{C}) = \mathsf{H}(\mathcal{C}) \cap \mathbf{OS}$.

Nechť $\mathcal{V}$ je třída struktur jazyka L. Budeme říkat, že $\mathcal{V}$ je varieta uspořádaných pologrup, budou-li splněny následující podmínky:

- $\mathcal{V} \subseteq \mathbf{OS}$
- $\mathsf{H}_{\mathbf{OS}}(\mathcal{V}) \subseteq \mathcal{V}$
- $\mathsf{S}(\mathcal{V}) \subseteq \mathcal{V}$
- $\mathsf{Pr}(\mathcal{V}) \subseteq \mathcal{V}$.

Nyní jedno důležité upozornění: V článku [3] se místo $\mathsf{H}_{\mathbf{OS}}$ píše pouze H. Je to dáno tím, že v celém článku [3] se ze struktur jazyka L berou v úvahu pouze uspořádané pologrupy.

Nyní je na místě zmínit pojem varieta uspořádaných algeber, který zavedl a zkoumal Bloom v [22].

Buď dán typ τ. Uspořádaná algebra typu τ je struktura A jazyka $\tau \cup \{\leq\}$, kde $\leq$ je binární relační symbol, $(A, \leq^A)$ je uspořádaná množina a pro všechna $* \in \tau$ platí: Jestliže $*$ má četnost n a $a_1, a_2, \ldots, a_n \in A$, $b_1, b_2, \ldots, b_n \in A$, pak

$$a_1 \leq^A b_1 \wedge a_2 \leq^A b_2 \wedge \cdots \wedge a_n \leq^A b_n \Longrightarrow *^A(a_1, a_2, \ldots, a_n) \leq^A *^A(b_1, b_2, \ldots, b_n).$$

Třídu všech uspořádaných algeber typu τ označme $\mathbf{O}$.

Pro $\mathcal{C} \subset \mathbf{O}$ položme $\mathsf{H}_{\mathbf{O}}(\mathcal{C}) = \mathsf{H}(\mathcal{C}) \cap \mathbf{O}$.

Bloom v [22] definuje operátor Q. Jeho definice je jiná, než naše definice operátoru $\mathsf{H}_{\mathbf{O}}$, avšak ekvivalentní (tj. pro každé $\mathcal{C} \subseteq \mathbf{O}$ je $\mathsf{Q}(\mathcal{C}) = \mathsf{H}_{\mathbf{O}}(\mathcal{C})$).

Dále Bloom definuje varietu uspořádaných algeber. Nechť $\mathcal{V}$ je třída struktur jazyka $\tau \cup \{\leq\}$. Budeme říkat, že $\mathcal{V}$ je varieta uspořádaných algeber typu τ, platí-li:

- $\mathcal{V} \subseteq \mathbf{O}$
- $\mathsf{H}_{\mathbf{O}}(\mathcal{V}) \subseteq \mathcal{V}$
- $\mathsf{S}(\mathcal{V}) \subseteq \mathcal{V}$
- $\mathsf{Pr}(\mathcal{V}) \subseteq \mathcal{V}$.

Variety uspořádaných pologrup jsou speciálním případem variet uspořádaných grupoidů.

Poznamenejme, že varietám uspořádaných algeber obecně se věnuje také Don Pigozzi v [43].

Vraťme se teď k naší analogii Věty 10 pro třídu $\mathbf{OS}$.

Vyjasníme, jaké relace budou hrát roli úplně invariantních kongruencí.

Nechť Y je neprázdná množina. Nerovnost je libovolná uspořádaná dvojice $u \preceq v$ slov $u, v \in Y^+$.

Nechť S je uspořádaná pologrupa. Nerovnost $u \preceq v$ je splněna v S (nebo S splňuje nerovnost $u \preceq v$), pokud $\varphi(u) \leq^S \varphi(v)$ pro každý homomorfismus $\varphi : Y^+ \to (S, \cdot^S)$. Nerovnost $u \preceq v$ je splněna ve třídě $\mathcal{C}$ uspořádaných pologrup, je-li splněna v každé uspořádané pologrupě z $\mathcal{C}$.

Pro danou třídu $\mathcal{C}$ uspořádaných pologrup označme $\mathsf{In}_Y(\mathcal{C})$ množinu všech uspořádaných dvojic $(u, v) \in Y^+ \times Y^+$ takových, že nerovnost $u \preceq v$ je splněna v $\mathcal{C}$.

Pro danou množinu nerovností $\Sigma \subseteq Y^+ \times Y^+$ označme $\mathsf{Mod}_Y(\Sigma)$ třídu všech uspořádaných pologrup S takových, že S splňuje všechny nerovnosti ze Σ.

Snadno se dokáže následující lemma.

Lemma 11 ([3], Lemma 2.1). *Nechť $\Sigma \subseteq Y^+ \times Y^+$. Pak $\mathsf{Mod}_Y(\Sigma)$ je varieta uspořádaných pologrup.*

Kvaziuspořádání (tj. reflexivní a tranzitivní relace) ρ na pologrupě S se nazývá stabilní, pokud pro všechna $a, b, c \in S$ platí: $a\rho b$ implikuje $ca\rho cb$ a $ac\rho bc$.

Nechť ρ je stabilní kvaziuspořádání na pologrupě S. Definujme binární relaci $\sim_\rho$ na S takto: pro $a, b \in S$,

$$a \sim_\rho b \iff a\rho b \wedge b\rho a.$$

Snadno lze ukázat, že $\sim_\rho$ je kongruence na pologrupě S. Kongruence $\sim_\rho$ určuje pologrupu $(S/\sim_\rho, \circ)$. Definujme na množině $S/\sim_\rho$ relaci $\ll$ takto: pro $a, b \in S$,

$$(a \sim_\rho) \ll (b \sim_\rho) \iff a\rho b.$$

Snadno zkontrolujeme, že relace $\ll$ na množině $S/\sim_\rho$ je definována korektně. Dále, $(S/\sim_\rho, \circ, \ll)$ je uspořádaná pologrupa. Budeme ji označovat S/ρ.

Kvaziuspořádaní ρ na pologrupě S se nazývá úplně invariantní, jestliže pro všechna $a, b \in S$ a každý endomorfismus $\eta : S \to S$ platí: $a\rho b$ implikuje $\eta(a)\rho\eta(b)$.

Množinu všech úplně invariantních stabilních kvaziuspořádání na pologrupě S budeme označovat $\mathsf{FISQ}(S)$.

Lemma 12 ([3], Lemma 2.3). *Nechť $\mathcal{C} \subseteq \mathbf{OS}$. Pak $\mathsf{In}_Y(\mathcal{C}) \in \mathsf{FISQ}(Y^+)$.*

Nyní přicházíme k důležité větě:

Věta 13 ([3], Theorem 2.5). *Nechť $\mathcal{V}$ je varieta uspořádaných pologrup. Pak $Y^+/\mathsf{In}_Y(\mathcal{V})$ je volná struktura v $\mathsf{Mod}_Y(\mathsf{In}_Y(\mathcal{V}))$ na množině Y a $Y^+/\mathsf{In}_Y(\mathcal{V}) \in \mathcal{V}$. Specielně, $Y^+/\mathsf{In}_Y(\mathcal{V})$ je volná struktura ve $\mathcal{V}$ na Y.*

Pro $\mathcal{C} \subseteq \mathbf{OS}$ je $\mathsf{I}(\mathcal{C}) \subseteq \mathsf{H}_{\mathbf{OS}}(\mathcal{C})$. Pro netriviální varietu $\mathcal{V}$ uspořádaných pologrup tedy existence volné struktury ve $\mathcal{V}$ na množině Y vyplývá z obecné Věty 8. Naše Věta 13 podává syntaktický popis volných struktur ve varietách uspořádaných pologrup pomocí úplně invariantních stabilních kvaziuspořádání na volných pologrupách.

Dospěli jsme k analogii Věty 10 pro třídu **OS**:

Věta 14 ([3], Theorem 2.10). *Pravidla*

$$\mathcal{V} \mapsto \mathsf{In}_X(\mathcal{V}), \;\; \rho \mapsto \mathsf{Mod}_X(\rho)$$

určují vzájemně inverzní bijekce mezi všemi varietami uspořádaných pologrup a všemi úplně invariantními stabilními kvaziuspořádáními na X^+*. Přitom*

1. *pro libovolné variety* $\mathcal{V}_1$*,* $\mathcal{V}_2$ *uspořádaných pologrup platí:*

$$\mathcal{V}_1 \subseteq \mathcal{V}_2 \Longleftrightarrow \mathsf{In}_X(\mathcal{V}_1) \supseteq \mathsf{In}_X(\mathcal{V}_2)$$

2. *pro libovolná úplně invariantní stabilní kvaziuspořádání* ρ_1*,* ρ_2 *na* X^+ *platí:*

$$\rho_1 \subseteq \rho_2 \Longleftrightarrow \mathsf{Mod}_X(\rho_1) \supseteq \mathsf{Mod}_X(\rho_2).$$

Důsledek 15. *Nechť* $\mathcal{C}$ *je třída uspořádaných pologrup. Platí:* $\mathcal{C}$ *je varieta uspořádaných pologrup právě tehdy, když* $\mathcal{C} = \mathsf{Mod}_X(\Sigma)$ *pro nějakou množinu nerovností* $\Sigma \subseteq X^+ \times X^+$*.*

Právě uvedený důsledek je analogií Birkhoffovy věty (viz Větu 9) pro třídy uspořádaných pologrup.

Obecněji, Bloom ve své prácí [22] formuluje a dokazuje analogii Birkhoffovy věty pro variety uspořádaných algeber daného typu τ (Theorem 2.6). Viz také [43], Theorem 3.14.

Naším přínosem v této oblasti je především Věta 14 udávající syntaktický popis variet uspořádaných pologrup pomocí úplně invariantních stabilních kvaziuspořádání na volné pologrupě X^+.

Nechť $\mathcal{V}$ je varieta (uspořádaných, polosvazově uspořádaných) pologrup. Všechny podvariety variety $\mathcal{V}$, uspořádané relací inkluze, tvoří svaz. Tento svaz označme $\mathcal{L}(\mathcal{V})$.

Nechť $\mathcal{C}$ je třída pologrup. Klademe

$$\mathbf{O}\mathcal{C} = \{(S, \cdot^S, \leq^S) \in \mathbf{OS} |\; (S, \cdot^S) \in \mathcal{C}\}.$$

Dále, nechť $\mathsf{Eq}_X(\mathcal{C})$ značí množinu všech uspořádaných dvojic $(u, v) \in X^+ \times X^+$ takových, že identita $u = v$ je splněna ve všech pologrupách ze třídy $\mathcal{C}$. Poznamenejme, že $\mathsf{Eq}_X(\mathcal{C})$ je totéž co $\mathsf{Eq}_{\mathbf{S}}(\mathcal{C})$ v označení použitém ve Větě 10.

Následující věta popisuje vztah mezi $\mathcal{L}(\mathbf{S})$ a $\mathcal{L}(\mathbf{OS})$.

Věta 16 ([3], Theorem 3.1). *Nechť* $\mathcal{V}$ *je varieta pologrup. Pak*

1. $\mathbf{O}\mathcal{V}$ *je varieta uspořádaných pologrup*

2. $\mathsf{In}_X(\mathbf{O}\mathcal{V}) = \mathsf{Eq}_X(\mathcal{V})$

3. *Zobrazení*

$$\mathbf{O} : \mathcal{L}(\mathcal{V}) \to \mathcal{L}(\mathbf{O}\mathcal{V})$$

je vnoření svazu všech podvariet variety $\mathcal{V}$ *do svazu všech podvariet variety* $\mathbf{O}\mathcal{V}$.

2.2 Variety uspořádaných bandů

Přijmeme následující označení pro variety pologrup (některé značení jsme zavedli již dříve):

- **T** triviální pologrupy (pologrupy splňující identitu $a = x$)
- **LZ** pologrupy levých nul ($ax = a$)
- **RZ** pologrupy pravých nul ($xa = a$)
- **SL** polosvazy ($x^2 = x, xy = yx$)
- **LNB** levé normální bandy ($x^2 = x, axy = ayx$)
- **RNB** pravé normální bandy ($x^2 = x, xya = yxa$)
- **ReB** rektangulární bandy ($x^2 = x, a = axa$)
- **NB** normální bandy ($x^2 = x, axya = ayxa$)
- **B** bandy ($x^2 = x$)
- **S** pologrupy.

V této části podáme popis svazu $\mathcal{L}(\mathbf{OB})$ všech variet uspořádaných bandů Tento popis je proveden (samozřejmě včetně důkazů) ve čtvrté části článku [3].

Popis svazu $\mathcal{L}(\mathbf{B})$ všech variet bandů podali roku 1970 Birjukov [21], Fennemore [34] a Gerhard [36] (viz také [37]).

Významný krok v popisu svazu $\mathcal{L}(\mathbf{OB})$ učinil Emery, který ve své práci [33] popsal svaz všech variet uspořádaných normálních bandů.

Použijeme zkratku: pro $u, v \in X^+$ píšeme pouze $u = v$ namísto $u \preceq v, v \preceq u$.

Věta 17 (S. J. Emery [33], Theorem 2.1). *Svaz* $\mathcal{L}(\mathbf{ONB})$ *všech variet uspořádaných normálních bandů sestává z následujících 16 variet (viz také obrázek 1):*

- $\mathcal{V}_1 = \mathsf{Mod}_X(x^2 = x, axya = ayxa) = \mathbf{ONB}$

- $\mathcal{V}_2 = \mathsf{Mod}_X(x^2 = x, axa \preceq a)$
- $\mathcal{V}_3 = \mathsf{Mod}_X(x^2 = x, a \preceq axa)$
- $\mathcal{V}_4 = \mathsf{Mod}_X(x^2 = x, a = axa) = \mathbf{OReB}$
- $\mathcal{V}_5 = \mathsf{Mod}_X(x^2 = x, axy = ayx) = \mathbf{OLNB}$
- $\mathcal{V}_6 = \mathsf{Mod}_X(x^2 = x, ax \preceq a)$
- $\mathcal{V}_7 = \mathsf{Mod}_X(x^2 = x, a \preceq ax)$
- $\mathcal{V}_8 = \mathsf{Mod}_X(x^2 = x, ax = a) = \mathbf{OLZ}$
- $\mathcal{V}_9 = \mathsf{Mod}_X(x^2 = x, xya = yxa) = \mathbf{ORNB}$
- $\mathcal{V}_{10} = \mathsf{Mod}_X(x^2 = x, xa \preceq a)$
- $\mathcal{V}_{11} = \mathsf{Mod}_X(x^2 = x, a \preceq xa)$
- $\mathcal{V}_{12} = \mathsf{Mod}_X(x^2 = x, xa = a) = \mathbf{ORZ}$
- $\mathcal{V}_{13} = \mathsf{Mod}_X(x^2 = x, xy = yx) = \mathbf{OSL}$
- $\mathcal{V}_{14} = \mathsf{Mod}_X(x^2 = x, xax \preceq a)$
- $\mathcal{V}_{15} = \mathsf{Mod}_X(x^2 = x, a \preceq xax)$
- $\mathcal{V}_{16} = \mathsf{Mod}_X(x^2 = x, a = x) = \mathbf{OT}$.

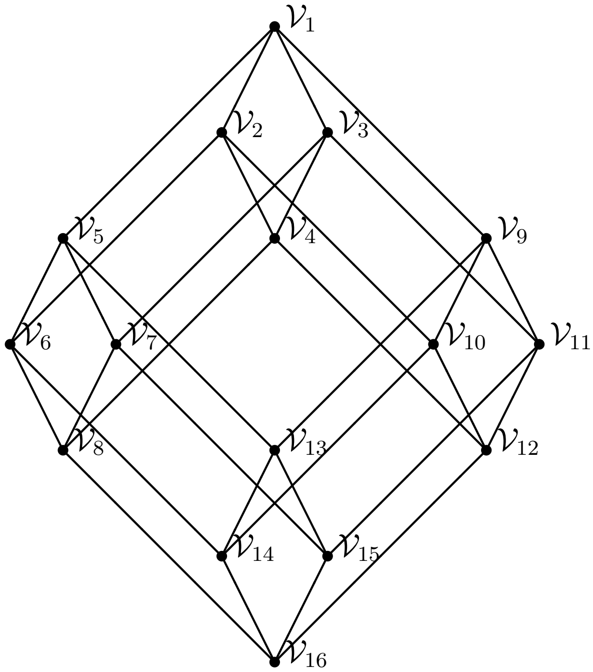
V [3] se podařilo dokázat následující dvě věty:

Věta 18 ([3], Theorem 4.6). *Zobrazení* $\mathbf{O} : [\mathbf{SL}, \mathbf{B}] \to [\mathbf{OSL}, \mathbf{OB}]$ *je izomorfismus.*

Věta 19 ([3], Theorem 4.9). *Jestliže* $\mathcal{V}$ *je varieta uspořádaných bandů,* $\mathbf{OSL} \not\subseteq \mathcal{V}$*, pak* $\mathcal{V} \subseteq \mathbf{ONB}$.

Uvědomme si nyní, že Věta 18 a Věta 19 poskytují úplný popis svazu $\mathcal{L}(\mathbf{OB})$. Proč tomu tak je? Svaz $\mathcal{L}(\mathbf{OB})$ se skládá z uzavřeného intervalu $[\mathbf{OSL}, \mathbf{OB}]$ a z variet $\mathcal{V} \in \mathcal{L}(\mathbf{OB})$ takových, že $\mathbf{OSL} \not\subseteq \mathcal{V}$. Jestliže $\mathcal{V}$ je varieta uspořádaných bandů, $\mathbf{OSL} \not\subseteq \mathcal{V}$, pak $\mathcal{V} \in \mathcal{L}(\mathbf{ONB})$ (viz Větu 19). Svaz $\mathcal{L}(\mathbf{ONB})$ všech variet uspořádaných normálních bandů plně popsal Emery [33] (zde Věta 17). Dále víme z Věty 18, že uzavřené intervaly $[\mathbf{SL}, \mathbf{B}]$, $[\mathbf{OSL}, \mathbf{OB}]$ jsou izomorfní (izomorfismus je dán zobrazením $\mathcal{V} \mapsto \mathbf{O}\mathcal{V}$). A konečně, svaz všech variet bandů (a tedy také interval $[\mathbf{SL}, \mathbf{B}]$) je plně popsán v [21], [34], [36].

Obrázek 1: Svaz $\mathcal{L}(\mathbf{ONB})$



Uvažme vnoření $\mathbf{O} : \mathcal{L}(\mathbf{S}) \to \mathcal{L}(\mathbf{OS})$, $\mathcal{V} \mapsto \mathbf{O}\mathcal{V}$. Nechť $M \subseteq \mathcal{L}(\mathbf{S})$. Položme, jak je zvykem,

$$\mathbf{O}(M) = \{\mathbf{O}\mathcal{V} | \ \mathcal{V} \in M\}.$$

Platí následující věta:

Věta 20 ([3], Theorem 4.12). *V označení z Věty 17,*

$$\mathcal{L}(\mathbf{OB}) - \mathbf{O}(\mathcal{L}(\mathbf{B})) = \{\mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_6, \mathcal{V}_7, \mathcal{V}_{10}, \mathcal{V}_{11}, \mathcal{V}_{14}, \mathcal{V}_{15}\}.$$

Poznamenejme na závěr této části, že popis svazu všech variet uspořádaných bandů využívají Almeida a Klíma ve své práci [17].

3 Variety polosvazově uspořádaných pologrup

3.1 Přípustné uzávěrové operátory

Je obecně známo, že v různých oblastech matematiky hrají významnou roli uzávěrové operátory.

Nechť M je množina. Zobrazení $[\ ] : \mathsf{P}(M) \to \mathsf{P}(M)$ se nazývá uzávěrový operátor na množině M, pokud pro všechna $A, B \subseteq M$ platí:

1. $A \subseteq [A]$

2. $[[A]] = [A]$

3. $A \subseteq B$ implikuje $[A] \subseteq [B]$.

Uzávěrový operátor [ ] na množině M se nazývá finitární (nebo také algebraický), pokud pro každé $A \subseteq M$ platí

$$[A] = \bigcup_{B \in \mathsf{P}^0_f(A)} [B]$$

Říkáme, že uzávěrový operátor [ ] zachovává prázdnou množinu, pokud $[\emptyset] = \emptyset$.

Klasickou oblastí, kde se pracuje s uzávěrovými operátory, je například topologie. Uzávěrový operátor [ ] na množině M je topologický uzávěrový operátor, pokud zachovává prázdnou množinu a pro všechna $A, B \subseteq M$ je $[A \cup B] = [A] \cup [B]$.

Nechť S je pologrupa, [ ] je uzávěrový operátor na S. Říkáme, že operátor [ ] je přípustný, jestliže pro všechna $A, B \subseteq S$ platí:

$$[A][B] \subseteq [AB]$$

Poznámka: Pro finitární uzávěrový operátor [ ] na S, který zachovává prázdnou množinu, stačí požadovat splnění podmínky $[A][B] \subseteq [AB]$ pouze pro všechny konečné neprázdné podmnožiny A, B množiny S.

Pojem přípustný uzávěrový operátor zavedl Jaromír Fuchs v [35] (Fuchs pracuje s uzávěrovými operátory na monoidech).

Samozřejmě, přípustný uzávěrový operátor je možno definovat nejen pro pologrupy, ale pro algebry libovolného typu.

Nechť A je algebra typu τ. Nechť $*$ je operační symbol jazyka τ, $*$ má četnost n (n je kladné celé číslo). Pro $B_1, \ldots, B_n \subseteq A$ klademe

$$*^{\mathsf{P}(A)}(B_1, \ldots, B_n) = \{*^A(b_1, \ldots, b_n) |\ b_1 \in B_1, \ldots, b_n \in B_n\}$$

Opět, nechť A je algebra typu τ. Nechť [ ] je uzávěrový operátor na A. Říkáme, že operátor [ ] je přípustný, jestliže pro každý operační symbol $*$ jazyka τ a pro všechna $B_1, \ldots, B_n$ (zde kladné celé číslo n značí četnost operačního symbolu $*$) platí:

$$*^{\mathsf{P}(A)}([B_1], \ldots, [B_n]) \subseteq [*^{\mathsf{P}(A)}(B_1, \ldots, B_n)]$$

Finitárními přípustnými uzávěrovými operátory zachovávajícími prázdnou množinu na algebrách obecného typu se zabývají Pilitowska a Zamojska-Dzienio v [46].

V části 3.1 habilitační práce budeme prezentovat výsledky článku [1], a to především částí 4, 5, 6. Stěžejním pojmem tohoto článku je ρ-přípustný uzávěrový operátor, kde $\rho \in \mathsf{FIC}(X^+)$ (přesná definice přijde za chvíli).

V [1] je ρ-přípustný uzávěrový operátor zaveden (přesněji: poprvé se tento pojem objevuje již v autorově diplomové práci [5]) a je ukázána jeho úzká souvislost s varietami polosvazově uspořádaných pologrup. Autor pak výsledky článku [1] použil mimo jiné ke studiu variet polosvazově uspořádaných normálních bandů (o tom více v části 3.2) a také k řešení problému identit pro variety polosvazově uspořádaných normální bandů (tomu je věnována část 3.3).

Řada autorů využívá výsledků článku [1] (případně je zobecňuje nebo aspoň považuje za důležité se o nich zmínit): [26], [30] (zde je citován rukopis článku [1]), [31], [44], [45], [46], [48], [49], [50], [51], [52], [53], [54], [55], [57], [58].

Snad lze dokonce říci, že vznik některých prací byl přímo motivován naším článkem [1]. Například v článku [46] jsou zobecněny metody prezentované v [1].

Přicházíme teď konečně k definici ρ-přípustného uzávěrového operátoru (viz [1], strany 37 a 38, nebo též [4], strana 38). Musíme však ještě zavést nějaké značení.

Pro $r \in X^*$ budeme symbolem $\mathsf{c}(r)$ označovat množinu všech proměnných slova r.

Pro $q \in X^+$, $x, y \in X$, definujeme $q(x|y)$ jako množinu všech slov získaných ze slova q dosazením y za některé výskyty proměnné x. Formálně: Jestliže $x \notin \mathsf{c}(q)$, pak $q(x|y) = \{q\}$. Jestliže $x \in \mathsf{c}(q)$, $q = q_1xq_2x \dots q_{k-1}xq_k$, kde $q_i \in X^*$, $x \notin \mathsf{c}(q_i)$ $(i = 1, \dots, k)$, pak

$$q(x|y) = \{q_1\}\{x,y\}\{q_2\}\{x,y\} \dots \{q_{k-1}\}\{x,y\}\{q_k\}$$

Uvedeme několik příkladů. Nechť x, y, z jsou tři různé proměnné. Pak
$x^2(x|y) = \epsilon x \epsilon x \epsilon = \{\epsilon\}\{x,y\}\{\epsilon\}\{x,y\}\{\epsilon\} = \{x^2, xy, yx, y^2\}$
$xz(x|y) = \epsilon xz(x|y) = \{\epsilon\}\{x,y\}\{z\} = \{xz, yz\}$
$xz(x|x) = \epsilon xz(x|x) = \{\epsilon\}\{x,x\}\{z\} = \{\epsilon\}\{x\}\{z\} = \{xz\}$

Nechť $\rho \in \mathsf{FIC}(X^+)$. Zobrazení $[\ ] : \mathsf{P}(X^+/\rho) \to \mathsf{P}(X^+/\rho)$ se nazývá ρ-přípustný uzávěrový operátor (nebo pouze přípustný operátor, je-li ρ známo z kontextu), jestliže je to finitární uzávěrový operátor na množině X^+/ρ, který zachovává prázdnou množinu a navíc splňuje následující podmínky:

(I) $[\{t\}] = [\{u\}]$ implikuje $t = u$

(II) $[T][U] \subseteq [TU]$

(III) $f([T]) \subseteq [f(T)]$

(IV) $q\rho \in [\{q_1\rho, \dots, q_k\rho\}]$ implikuje $q(x|y)\rho \subseteq [q_1(x|y)\rho \cup \dots \cup q_k(x|y)\rho]$

pro všechna $t, u \in X^+/\rho$, $T, U \in \mathsf{P}_f(X^+/\rho)$, $f \in \mathsf{End}(X^+/\rho)$, $q, q_1, \dots, q_k \in X^+$, $x, y \in X$.

Přitom pro $M \subseteq X^+$ je $M\rho = \{q\rho |\ q \in M\}$.

Řadu příkladů explicitně zadaných ρ-přípustných uzávěrových operátorů lze najít v části 3.2 (konkrétně všechny ρ-přípustné uzávěrové operátory pro $\rho \in \{\rho_{\mathbf{T}}, \rho_{\mathbf{LZ}}, \rho_{\mathbf{SL}}, \rho_{\mathbf{LNB}}\}$).

Nechť $\mathcal{C}$ je třída pologrup. Označíme $\mathbf{SLO}\mathcal{C}$ třídu všech polosvazově uspořádaných pologrup $(S, \cdot, \vee)$ takových, že $(S, \cdot) \in \mathcal{C}$. Jestliže $\mathcal{V}$ je varieta pologrup, pak $\mathbf{SLO}\mathcal{V}$ je varieta polosvazově uspořádaných pologrup.

Víme, že $X^\Box$ je volná algebra na množině X ve varietě **SLOS** (viz Větu 2). Dle Věty 10 pak existuje bijekce mezi všemi podvarietami variety **SLOS** a všemi úplně invariantními kongruencemi polosvazově uspořádané pologrupy $X^\Box$.

Nechť $\sim \in \mathsf{FIC}(X^\Box)$. Definujeme binární relaci $\rho_\sim$ na X^+ následujícím způsobem: pro $q, r \in X^+$,

$$q \rho_\sim r \iff \{q\} \sim \{r\}$$

Zřejmě $\rho_\sim$ je kongruence pologrupy X^+. Dále definujeme operátor $[\,]_\sim : \mathsf{P}(X^+/\rho_\sim) \to \mathsf{P}(X^+/\rho_\sim)$ následovně: pro $T \in \mathsf{P}(X^+/\rho_\sim)$, $q \in X^+$,

$q\rho_\sim \in [T]_\sim$

právě tehdy, když

$\{q_1, \dots, q_k, q\} \sim \{q_1, \dots, q_k\}$ pro nějaké kladné celé číslo k a $q_1, \dots, q_k \in X^+$ taková, že $q_1\rho_\sim, \dots, q_k\rho_\sim \in T$.

Všimněme si, že definice operátoru $[\ ]_\sim$ je korektní: Nechť $q, r, q_1, \dots, q_k \in X^+$, $q\rho_\sim r$, $\{q_1, \dots, q_k, q\} \sim \{q_1, \dots, q_k\}$, $q_1\rho_\sim, \dots, q_k\rho_\sim \in T$. Jelikož $\{q\} \sim \{r\}$, dostáváme $\{q_1, \dots, q_k, q\} \sim \{q_1, \dots, q_k, r\}$ a tedy $\{q_1, \dots, q_k, r\} \sim \{q_1, \dots, q_k\}$.

Nyní můžeme zformulovat hlavní větu z [1].

Věta 21 ([1], Theorem 4.7).

(i) *Nechť* $\sim \in \mathsf{FIC}(X^\Box)$. *Pak* $\rho_\sim \in \mathsf{FIC}(X^+)$ *a* $[\ ]_\sim$ *je* $\rho_\sim$*-přípustný uzávěrový operátor.*

(ii) *Nechť* $\rho \in \mathsf{FIC}(X^+)$ *a nechť* $[\,]$ *je* ρ*-přípustný uzávěrový operátor. Definujme binární relaci* $\sim_{\rho,[\ ]}$ *na* $\mathsf{P}_f(X^+)$ *následovně: pro* $Q, R \in \mathsf{P}_f(X^+)$ *položíme*

$$Q \sim_{\rho,[\ ]} R \iff [Q\rho] = [R\rho].$$

Pak $\sim_{\rho,[\ ]} \in \mathsf{FIC}(X^\Box)$.

(iii) *Pro všechna* $\sim \in \mathsf{FIC}(X^\Box)$ *platí:* $\sim \,=\, \sim_{\rho_\sim,[\ ]_\sim}$.

(iv) *Pro všechna* $\rho \in \mathsf{FIC}(X^+)$ *a všechny* ρ*-přípustné uzávěrové operátory* $[\,]$ *platí:*

$$\rho = \rho_{\sim_{\rho,[\ ]}},\ \ [\,] = [\,]_{\sim_{\rho,[\ ]}}.$$

Existuje tedy bijekce mezi všemi úplně invariantními kongruencemi polosvazově uspořádané pologrupy $X^\Box$ a všemi uspořádanými dvojicemi $(\rho, [\,])$, kde ρ je úplně invariantní kongruence pologrupy X^+ a $[\,]$ je ρ-přípustný uzávěrový operátor. Tato bijekce je dána přiřazeními

$$\sim \;\mapsto (\rho_\sim, [\,]_\sim),\ \ (\rho, [\,]) \mapsto \;\sim_{\rho,[\ ]}$$

Uveďme ještě větu popisující průnik úplně invariantních kongruencí polosvazově uspořádané pologrupy $X^{\square}$ pomocí přípustných uzávěrových operátorů.

Věta 22 ([1], Theorem 4.9). *Nechť $\rho_1, \rho_2 \in \mathsf{FIC}(X^+)$, $[\ ]_1$ je ρ_1-přípustný uzávěrový operátor, $[\ ]_2$ je ρ_2-přípustný uzávěrový operátor. Položme $\sim\ = \ \sim_{\rho_1,[\ ]_1} \cap \sim_{\rho_2,[\ ]_2}$. Pak*

$$\rho_\sim = \rho_1 \cap \rho_2$$

$$q\rho_\sim \in [\{q_1\rho_\sim, \dots, q_k\rho_\sim\}]_\sim \iff \begin{array}{l} q\rho_1 \in [\{q_1\rho_1, \dots, q_k\rho_1\}]_1, \\ q\rho_2 \in [\{q_1\rho_2, \dots, q_k\rho_2\}]_2 \end{array}$$

$(q, q_1, \dots, q_k \in X^+)$.

Operátor $[\ ]_\sim$ z Věty 22 označíme $[\ ]_1 \wedge [\ ]_2$. Při tomto označení můžeme psát

$$\sim_{\rho_1,[\ ]_1} \cap \sim_{\rho_2,[\ ]_2} = \sim_{\rho_1\cap\rho_2,[\ ]_1\wedge[\ ]_2}$$

Nechť $\mathcal{V}$ je varieta pologrup.

Pak $\mathbf{SLO}\mathcal{V}$ je varieta určená množinou identit $\{(\{u\},\{v\}) \mid (u,v) \in \rho_{\mathcal{V}}\}$. Nechť $\mathcal{W}$ je varieta polosvazově uspořádaných pologrup , $\sim$ je odpovídající úplně invariantní kongruence polosvazově uspořádané pologrupy $X^{\square}$. Dle Věty 21 $\sim$ odpovídá uspořádané dvojici $(\rho, [\ ])$, kde $\rho \in \mathsf{FIC}(X^+)$ a $[\ ]$ je ρ-přípustný uzávěrový operátor. Máme:

$$\begin{aligned} \mathcal{W} \subseteq \mathbf{SLO}\mathcal{V} &\iff \sim \supseteq \{(\{u\},\{v\}) \mid (u,v) \in \rho_{\mathcal{V}}\} \\ &\iff \rho \supseteq \rho_{\mathcal{V}} \end{aligned}$$

Tedy: Nalezení všech podvariet variety $\mathbf{SLO}\mathcal{V}$ je redukováno na popis všech ρ-přípustných uzávěrových operátorů pro všechna $\rho \in \mathsf{FIC}(X^+)$ taková, že $\rho \supseteq \rho_{\mathcal{V}}$.

Nám se podařilo najít všechny ρ-přípustné uzávěrové operátory pro $\rho \in \mathsf{FIC}(X^+)$ taková, že $\rho \supseteq \rho_{\mathbf{LNB}}$. Tím je získán popis svazu $\mathcal{L}(\mathbf{SLOLNB})$ pomocí přípustných uzávěrových operátorů (viz následující část 3.2). Tento výsledek pak dává dva další: popis svazu $\mathcal{L}(\mathbf{SLONB})$ pomocí přípustných uzávěrových operátorů a také řešení problému identit pro všechny variety polosvazově uspořádaných normálních bandů (viz 3.3).

3.2 Variety polosvazově uspořádaných normálních bandů

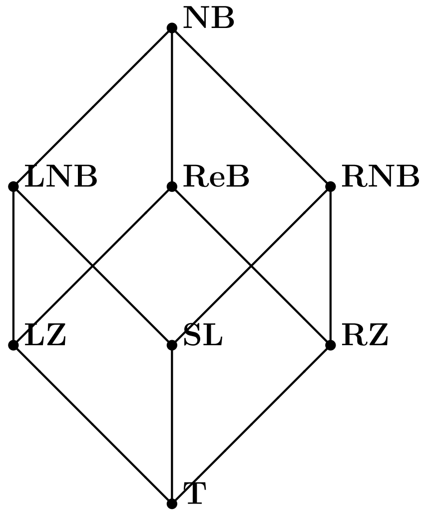
Nyní vyložíme hlavní výsledky článku [4].

Na obrázku 2 vidíme svaz všech podvariet variety **NB**.

V [4] jsou nejprve explicitně popsány všechny ρ-přípustné uzávěrové operátory, kde $\rho \supseteq \rho_{\mathbf{LNB}}$.

Připomeňme popis volných algeber na množině X v některých podvarietách variety **NB**. Nejprve vysvětlíme označení: Nechť $w \in X^+$. Pak $\mathsf{c}(w)$ je množina všech proměnných (písmen) slova w (toto označení jsme zavedli již dřív), $\mathsf{h}(w)$ je první proměnná (písmeno) slova w, $\mathsf{t}(w)$ je poslední proměnná (písmeno) slova w. Pro libovolná $u, v \in X^+$ máme

Obrázek 2: Svaz $\mathcal{L}(\mathbf{NB})$



(i) $u\rho_{\mathbf{LZ}}v$ právě tehdy, když $\mathsf{h}(u) = \mathsf{h}(v)$
$X^+/\rho_{\mathbf{LZ}} \cong (X, \circ)$, kde $x \circ y = x$

(ii) $u\rho_{\mathbf{SL}}v$ právě tehdy, když $\mathsf{c}(u) = \mathsf{c}(v)$
$X^+/\rho_{\mathbf{SL}} \cong (\mathsf{P}_f(X), \cup)$

(iii) $u\rho_{\mathbf{LNB}}v$ právě tehdy, když $\mathsf{c}(u) = \mathsf{c}(v), \mathsf{h}(u) = \mathsf{h}(v)$
$X^+/\rho_{\mathbf{LNB}} \cong (\{(y, Y)|\ Y \in \mathsf{P}_f(X), y \in Y\}, \circ)$ kde $(y, Y) \circ (z, Z) = (y, Y \cup Z)$, a izomorfismus je dán předpisem $u\rho_{\mathbf{LNB}} \mapsto (\mathsf{h}(u), \mathsf{c}(u))$

(iv) $u\rho_{\mathbf{RNB}}v$ právě tehdy, když $\mathsf{c}(u) = \mathsf{c}(v), \mathsf{t}(u) = \mathsf{t}(v)$
$X^+/\rho_{\mathbf{RNB}} \cong (\{(Y, y)|\ Y \in \mathsf{P}_f(X), y \in Y\}, \circ)$ kde $(Y, y) \circ (Z, z) = (Y \cup Z, z)$, a izomorfismus je dán předpisem $u\rho_{\mathbf{RNB}} \mapsto (\mathsf{c}(u), \mathsf{t}(u))$

Jak je častým zvykem, izomorfní struktury budeme ztotožňovat - tedy místo $X^+/\rho_{\mathbf{LZ}}$ budeme pracovat s $(X, \circ)$, a podobně. Doufejme přitom, že nedojde k nedorozumění.

Existuje přesně jeden uzávěrový operátor na jednoprvkové množině, který zachovává prázdnou množinu, totiž identický operátor. Platí tedy

Věta 23. *Existuje přesně jeden $\rho_{\mathbf{T}}$-přípustný uzávěrový operátor, totiž identitický operátor na* $\mathsf{P}(X^+/\rho_{\mathbf{T}})$.

Věta 24 ([5], Věta 5.2). *Existuje přesně jeden $\rho_{\mathbf{LZ}}$-přípustný uzávěrový operátor, totiž identický operátor na* $\mathsf{P}(X^+/\rho_{\mathbf{LZ}})$.

Definujeme čtyři operátory na $\mathsf{P}(X^+/\rho_{\mathbf{SL}})$. Nechť $T \in \mathsf{P}(X^+/\rho_{\mathbf{SL}})$, $Y \in X^+/\rho_{\mathbf{SL}}$. Klademe

$$
\begin{array}{lcl}
Y \in [T]_1 & \iff & Z \subseteq Y \text{ pro } Z \in T \\
Y \in [T]_2 & \iff & Y \subseteq \bigcup_{Z \in T} Z \\
Y \in [T]_3 & \iff & Z \subseteq Y \subseteq \bigcup_{Z \in T} Z \text{ pro } Z \in T \\
Y \in [T]_4 & \iff & Y = Z_1 \cup \cdots \cup Z_k \text{ pro } Z_1, \ldots, Z_k \in T
\end{array}
$$

Věta 25 ([5], Věta 5.3). *Existují přesně čtyři $\rho_{\mathbf{SL}}$-přípustné uzávěrové operátory, totiž* $[\ ]_1$, $[\ ]_2$, $[\ ]_3$, $[\ ]_4$.

Dále definujeme několik operátorů na $\mathsf{P}(X^+/\rho_{\mathbf{LNB}})$. Nechť $T \in \mathsf{P}(X^+/\rho_{\mathbf{LNB}})$, $(y, Y) \in X^+/\rho_{\mathbf{LNB}}$. Klademe

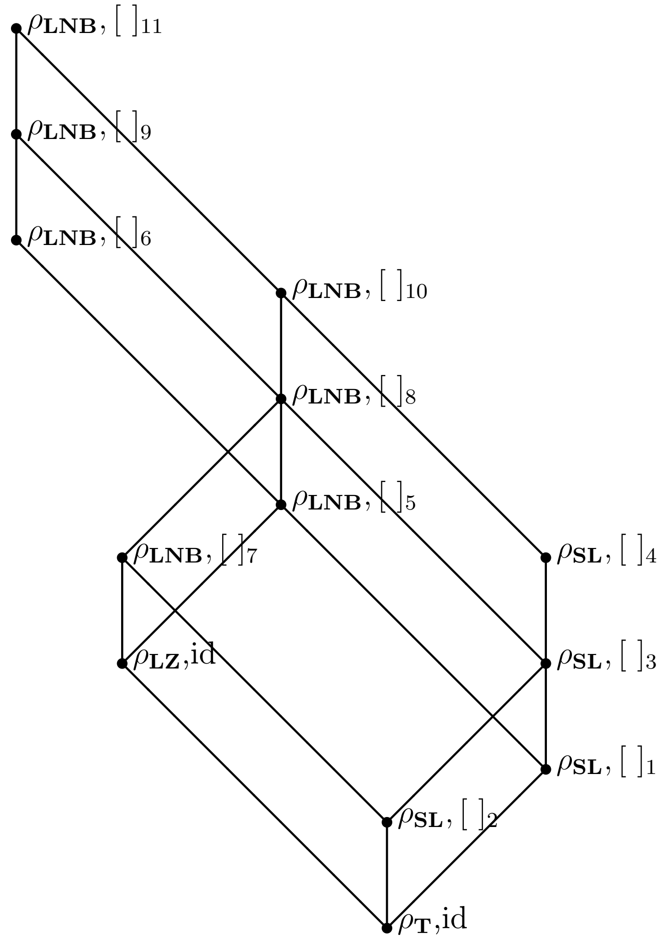
$$
\begin{array}{lcl}
(y, Y) \in [T]_5 & \iff & y = z_1, Z_2 \subseteq Y \text{ pro } (z_1, Z_1), (z_2, Z_2) \in T \\
(y, Y) \in [T]_6 & \iff & y = z, Z \subseteq Y \text{ pro } (z, Z) \in T \\
(y, Y) \in [T]_7 & \iff & y = z, Y \subseteq \bigcup_{(z,Z) \in T} Z \text{ pro } (z, Z) \in T \\
(y, Y) \in [T]_8 & \iff & y = z_1, Z_2 \subseteq Y \subseteq \bigcup_{(z,Z) \in T} Z \text{ pro } (z_1, Z_1), (z_2, Z_2) \in T \\
(y, Y) \in [T]_9 & \iff & y = z, Z \subseteq Y \subseteq \bigcup_{(z,Z) \in T} Z \text{ pro } (z, Z) \in T \\
(y, Y) \in [T]_{10} & \iff & y = z, Y = Z_1 \cup \cdots \cup Z_k \text{ pro } (z, Z), (z_1, Z_1), \ldots, (z_k, Z_k) \in T \\
(y, Y) \in [T]_{11} & \iff & y = z_1, Y = Z_1 \cup \cdots \cup Z_k \text{ pro } (z_1, Z_1), \ldots, (z_k, Z_k) \in T
\end{array}
$$

Věta 26 ([4], Theorem 4.6). *Existuje přesně sedm $\rho_{\mathbf{LNB}}$-přípustných uzávěrových operátorů, totiž* $[\ ]_5, [\ ]_6, [\ ]_7, [\ ]_8, [\ ]_9, [\ ]_{10}, [\ ]_{11}$.

Platí: $\rho \in \mathsf{FIC}(X^+)$, $\rho \supseteq \rho_{\mathbf{LNB}}$ právě tehdy, když $\rho \in \{\rho_{\mathbf{T}}, \rho_{\mathbf{LZ}}, \rho_{\mathbf{SL}}, \rho_{\mathbf{LNB}}\}$. Ve větách 23, 24, 25 a 26 jsou explicitně popsány všechny ρ-přípustné uzávěrové operátory pro $\rho \in \{\rho_{\mathbf{T}}, \rho_{\mathbf{LZ}}, \rho_{\mathbf{SL}}, \rho_{\mathbf{LNB}}\}$. Dostáváme tak popis všech variet polosvazově uspořádaných levých normálních bandů pomocí uspořádaných dvojic $(\rho, [\ ])$, kde $\rho \in \mathsf{FIC}(X^+)$, $[\ ]$ je ρ-přípustný uzávěrový operátor. S využitím Věty 22 dostáváme dokonce popis svazu $\mathcal{L}(\mathbf{SLOLNB})$:

Věta 27 ([4], Theorem 5.1) *Svaz všech variet polosvazově uspořádaných levých normálních bandů je distributivní, má 13 prvků a je uveden na obrázku 3.*

Obrázek 3: Svaz $\mathcal{L}(\mathbf{SLOLNB})$



Definujme ještě operátor na $\mathsf{P}(X^+/\rho_{\mathbf{RNB}})$. Nechť $T \in \mathsf{P}(X^+/\rho_{\mathbf{RNB}})$, $(Y,y) \in X^+/\rho_{\mathbf{RNB}}$. Klademe

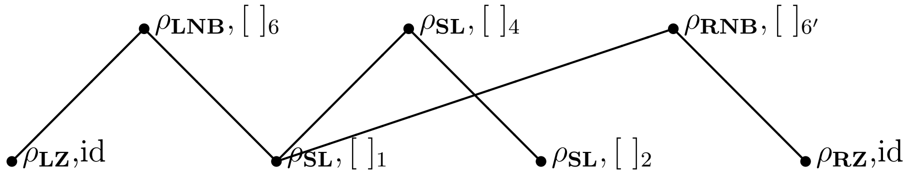
$$(Y,y) \in [T]_{6'} \Longleftrightarrow y = z, Z \subseteq Y \text{ pro } (Z,z) \in T$$

Z duální verze Věty 26 plyne, že $[\ ]_{6'}$ je $\rho_{\mathbf{RNB}}$-přípustný uzávěrový operátor.

Označme Irr množinu všech spojově ireducibilních variet v $\mathcal{L}(\mathbf{SLONB})$. Platí

Tvrzení 28 ([4], Lemma 6.1). *Svaz všech variet polosvazově uspořádaných normálních bandů obsahuje 7 spojově ireducibilních variet. Uspořádaná množina* $(Irr, \subseteq)$ *je na obrázku 4.*

Obrázek 4: Uspořádaná množina $(Irr, \subseteq)$



Není obtížné dokázat, že svaz $\mathcal{L}(\mathbf{SLONB})$ všech variet polosvazově uspořádaných normálních bandů je konečný a distributivní (v [41] je dokonce ukázáno, že svaz $\mathcal{L}(\mathbf{SLOB})$ je distributivní a má přesně 78 variet). Jsou tedy variety polosvazově uspořádaných normálních bandů ve vzájemně jednoznačné korespondenci s dědičnými podmnožinami uspořádané množiny $(Irr, \subseteq)$. Přitom dědičné podmnožině $\{\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_k\}$ odpovídá varieta $\mathcal{W}_1 \vee \mathcal{W}_2 \vee \cdots \vee \mathcal{W}_k$.

Je snadné najít všechny dědičné podmnožiny uspořádané množiny $(Irr, \subseteq)$. Existuje 1 prázdná dědičná podmnožina, 4 dědičné podmnožiny s jedním prvkem, 6 dědičných podmnožin se dvěma prvky, 7 dědičných podmnožin se třemi prvky, 7 dědičných podmnožin se čtyřmi prvky, 6 dědičných podmnožin s pěti prvky, 3 dědičné podmnožiny s šesti prvky a 1 dědičná podmnožina se sedmi prvky.

Existuje tedy přesně $1 + 4 + 6 + 7 + 7 + 6 + 3 + 1 = 35$ variet polosvazově uspořádaných normálních bandů.

Svaz $\mathcal{L}(\mathbf{SLONB})$ popsali Ghosh, Pastijn a Zhao v [38], Theorem 4.9. Naskýtá se tedy přirozená otázka: Jaký význam má naše zkoumání variet polosvazově uspořádaných normálních bandů? Na tuto otázku lze odpovědět následovně:

1. Je to ukázka "praktického" využití přípustných uzávěrových operátorů. Všechny variety polosvazově uspořádaných levých normálních bandů jsme našli tak, že jsme našli všechny

ρ-přípustné uzávěrové operátory pro $\rho \in \mathsf{FIC}(X^+)$, $\rho \supseteq \rho_{\mathbf{LNB}}$. Duálně jsme tedy také našli všechny variety polosvazově uspořádaných pravých normálních bandů. Tím jsme též našli všechny spojově ireducibilní variety ve svazu všech variet polosvazově uspořádaných normálních bandů a díky distributivitě tohoto svazu už vlastně všechny variety polosvazově uspořádaných normálních bandů. Stručně řečeno, nalezení všech ρ-přípustných uzávěrových operátorů pro $\rho \in \mathsf{FIC}(X^+)$, $\rho \supseteq \rho_{\mathbf{LNB}}$, poskytlo popis svazu $\mathcal{L}(\mathbf{SLONB})$. Variety polosvazově uspořádaných normálních bandů byly nalezeny jinak (jinou metodou) než v [38].

2. Variety polosvazově uspořádaných normálních bandů popisujeme pomocí uspořádaných dvojic $(\rho, [\ ])$, kde $\rho \in \mathsf{FIC}(X^+)$, $[\ ]$ je ρ-přípustný uzávěrový operátor. Tedy jinak, než v [38]. Jak uvidíme v další části, tento popis také dává řešení problému identit pro všechny variety polosvazově uspořádaných normálních bandů.

3. Za zmínku snad stojí také to, že již roku 1988 v [5] byla vyslovena hypotéza, že svaz variet polosvazově uspořádaných normálních bandů má 35 prvků.

3.3 Problém identit pro variety polosvazově uspořádaných normálních bandů

Obecně řečeno, varieta $\mathcal{V}$ algeber určitého typu má řešitelný problém identit, když existuje algoritmus rozhodující o každé identitě, zda platí ve $\mathcal{V}$.

Specifikujme problém identit pro variety polosvazově uspořádaných pologrup.

Dle Věty 2 je $X^{\square}$ volná algebra na množině X ve varietě všech polosvazově uspořádaných pologrup. Dle Věty 10 pravidla

$$\mathcal{W} \mapsto \mathsf{Eq}_{\mathbf{SLOS}}(\mathcal{W}),\ \sim\ \mapsto \mathsf{Mod}_{\mathbf{SLOS}}(\sim)$$

určují vzájemně inverzní bijekce mezi všemi podvarietami variety **SLOS** a všemi úplně invariantními kongruencemi algebry $X^{\square}$.

Nechť $\mathcal{V}$ je varieta polosvazově uspořádaných pologrup. Nechť $\sim\ \in \mathsf{FIC}(X^{\square})$, $\sim$ odpovídá varietě $\mathcal{V}$ (tj. $\sim\ = \mathsf{Eq}_{\mathbf{SLOS}}(\mathcal{V})$). Vyřešit problém identit pro varietu $\mathcal{V}$ znamená najít algoritmus, který pro každou dvojici $(Q, R) \in \mathsf{P}_f(X^+)$ rozhoduje, zda $Q \sim R$.

Je $\mathcal{V} = \mathcal{W}_1 \vee \mathcal{W}_2 \vee \cdots \vee \mathcal{W}_k$, kde $\{\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_k\}$ je dědičná podmnožina uspořádané množiny $(Irr, \subseteq)$. Nechť pro $i \in \{1, 2, \ldots, k\}$ je $\sim_i \in \mathsf{FIC}(X^{\square})$, $\sim_i$ odpovídá varietě $\mathcal{W}_i$. Je $\sim\ = \sim_1 \cap \sim_2 \cap \cdots \cap \sim_k$. Na základě Tvrzení 28 $\sim_i\ = \sim_{\sigma_i, \langle\ \rangle_i}$, kde $(\sigma_i, \langle\ \rangle_i)$ patří do množiny

$$\{(\rho_{\mathbf{LZ}}, \mathrm{id}), (\rho_{\mathbf{SL}}, [\ ]_1), (\rho_{\mathbf{SL}}, [\ ]_2), (\rho_{\mathbf{RZ}}, \mathrm{id}), (\rho_{\mathbf{LNB}}, [\ ]_6), (\rho_{\mathbf{SL}}, [\ ]_4), (\rho_{\mathbf{RNB}}, [\ ]_{6'})\}.$$

Nechť $(Q, R) \in \mathsf{P}_f(X^+)$. Platí:

$$\begin{aligned} Q \sim R \iff & Q \sim_{\sigma_i, \langle\ \rangle_i} R \text{ pro } i = 1, 2, \ldots, k \\ \iff & \langle Q\sigma_i \rangle_i = \langle R\sigma_i \rangle_i \text{ pro } i = 1.2, \ldots, k. \end{aligned}$$

Je $Q\sigma_i, R\sigma_i \in \mathsf{P}_f(X^+/\sigma_i)$ a v konečně mnoha krocích rozhodneme, zda $\langle Q\sigma_i \rangle_i = \langle R\sigma_i \rangle_i$. Umíme tedy řešit problém identit pro každou varietu $\mathcal{V}$ polosvazově uspořádaných normálních bandů.

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Separáty článků

Následují separáty článků v tomto pořadí:

1. Petr Gajdoš, Martin Kuřil: *Ordered semigroups of size at most 7 and linearly ordered semigroups of size at most 10*

2. Martin Kuřil: *On varieties of ordered semigroups*

3. Martin Kuřil, Libor Polák: *On varieties of semilattice-ordered semigroups*

4. Martin Kuřil: *Admissible closure operators and varieties of semilattice-ordered normal bands*

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RESEARCH ARTICLE

Ordered semigroups of size at most 7 and linearly ordered semigroups of size at most 10

Petr Gajdoš · Martin Kuřil

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Abstract We construct, classify and enumerate all non-isomorphic ordered semigroups of size at most 7. In the special case of linearly ordered semigroups we achieve size 10.

Keywords Ordered semigroup · Ordered set · Isomorphism

1 Introduction

A relation $\leq$ on a semigroup S is *stable* if, for every $a, b, c \in S$, $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$. An *ordered semigroup* is a semigroup S equipped with a stable partial order $\leq$ on S.

Let S and T be two ordered semigroups. A *morphism of ordered semigroups* $\varphi : S \to T$ is an order-preserving semigroup morphism from S into T.

Let $\mathcal{A}$, $\mathcal{B}$ be structures of the same type. If there exists an isomorphism from $\mathcal{A}$ to $\mathcal{B}$ then we say that $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic* and write $\mathcal{A} \cong \mathcal{B}$.

Let S and T be ordered semigroups. It follows directly from the definitions that $S \cong T$ implies S, T are isomorphic as semigroups and also S, T are isomorphic as ordered sets. The converse is not generally true.

A basic introduction to the theory of ordered semigroups is provided in [1].

We have constructed all non-isomorphic ordered semigroups with at most 7 elements. Obtained ordered semigroups have been classified and enumerated. Up to now,

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P. Gajdoš · M. Kuřil ( )
Katedra matematiky PřF UJEP, České mládeže 8, 400 96 Ústí nad Labem, Czech Republic
e-mail: martin.kuril@ujep.cz

P. Gajdoš
e-mail: pgajdos@suse.cz

ordered semigroups were enumerated only in two extreme cases: discretely ordered semigroups and linearly ordered semigroups. We are interested in semigroups which are ordered arbitrarily.

The problem of construction, classification and enumeration of discretely ordered semigroups is equivalent to the same problem for semigroups without order relation. Many authors have studied the problem (note that frequently two semigroups are considered to be equivalent if they are isomorphic or anti-isomorphic): Plemmons in [11], [12] (all non-equivalent semigroups of size n, for $n \leq 6$), Jürgensen and Wick in [9] (for $n \leq 7$), Satoh et al. in [15] (for $n \leq 8$), Distler in [3] (for $n \leq 9$), and others.

A few authors studied the problem of construction and enumeration of linearly ordered semigroups: Gabovich in [5] (for $n = 4$), Jürgensen in [8] (for $n \leq 6$), Slaney in [16] (for $n \leq 7$).

Now, some words about the structure of the paper. Section 2 has a preparatory character. We present facts concerning ordered sets that will be used later.

Section 3 forms the central part of our paper. We construct, classify and enumerate all non-isomorphic ordered semigroups with n elements, $n \leq 7$. First, in Sect.3.1, we study ordered 2-nilpotent semigroups. This is an example to demonstrate the difference to the enumeration of semigroups without order. In Sect. 3.2 we formulate an algorithm based on the ideas described by Plemmons in [12]. Using an implementation of the algorithm we have constructed all non-isomorphic ordered semigroups of size n, $n \leq 7$. Results of our computations are presented in Sect. 3.3, but we do not present the constructed semigroups here. The reason is simple—there are too many of them. We present only quantitative results of the classification (Table 3, Table 4). In the case of linearly ordered semigroups (Sect. 3.4) we were able to perform the computation for $n \leq 10$ (results are given in Table 5).

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2 Ordered sets

The section contains the information on the representation of ordered sets in the paper and on the computation of their automorphism groups.

For a positive integer n, let $G(n)$ denote the number of all non-isomorphic ordered sets with n elements. The values, presented in Table 1, can be found in [2] (see also [17] , Sequence A000112). Note that Brinkmann and McKay computed in [2] the values $G(n)$ for $n \leq 16$.

Table 1 Numbers of non-isomorphic ordered sets

n	1	2	3	4	5	6	7	8	9	10	11	12
$G(n)$	1	2	5	16	63	318	2045	16999	183231	2567284	46749427	1104891746

We need a suitable representation of ordered sets. Let n be a positive integer. Let $(E, \preceq)$ be an ordered set with n elements. Let us choose a bijection $f : \{1, 2, \ldots, n\} \to E$ such that for all $i, j \in \{1, 2, \ldots, n\}$, $f(i) \preceq f(j)$ implies $i \leq j$. Such a bijection exists since $\preceq$ has a linear extension. We define a binary matrix $A = (a_{ij})$ of size $n \times n$ in the following way:

$$a_{ij} = \begin{cases} 1 & f(i) \preceq f(j) \\ 0 & f(i) \not\preceq f(j) \end{cases}$$

for any $i, j \in \{1, 2, \ldots, n\}$.

Knowing the matrix A and the function f we know the relation $\preceq$ on the set E. If we identify $f(i)$ and i (i.e. we denote the element $f(i)$ by i), then $E = \{1, 2, \ldots, n\}$ and $i \preceq j$ implies $i \leq j$ and

$$a_{ij} = \begin{cases} 1 & i \preceq j \\ 0 & i \not\preceq j \end{cases}$$

for any $i, j \in \{1, 2, \ldots, n\}$.

Note that all elements of the matrix A lying under the main diagonal are equal to 0. Of course, all elements of the matrix A lying on the main diagonal are equal to 1 (the relation $\preceq$ is reflexive). Consequently, the complete information about the relation $\preceq$ is contained in a binary sequence of length $\frac{n^2-n}{2}$, namely in the sequence

$$a_{12} \ldots a_{1n} \cdot a_{23} \ldots a_{2n} \ldots a_{n-2,n-1} a_{n-2,n} . a_{n-1,n} .$$

The notation as a binary sequence mainly serves the purpose to specify the ordered set in a compact form, which is suitable for Table 3.

In order to make the sequences easy to read we put a dot between a_{in} and $a_{i+1,i+2}$, for $1 \leq i \leq n - 2$.

It should be mentioned that different sequences can represent isomorphic ordered sets. For example, the sequences 00.1, 01.0, 10.0 represent the ordered set consisting of one incomparable element and one 2-element chain .

Let E be an ordered set. An isomorphism from E to E is called an *automorphism* of the ordered set E. The set of all automorphisms of E will be denoted by $\mathrm{Aut}(E)$. The set $\mathrm{Aut}(E)$ is a subgroup in the symmetric group $\mathrm{Sym}(E)$ of the set E. If ordered sets E_1 and E_2 are isomorphic, then groups $\mathrm{Aut}(E_1)$ and $\mathrm{Aut}(E_2)$ are also isomorphic. If $\{1, 2, \ldots, n\}$, where n is a positive integer, is the underlying set of E then the automorphism group $\mathrm{Aut}(E)$ forms a subgroup in the symmetric group $\mathrm{Sym}(n)$. Two extreme cases follow.

1. If E is a discretely ordered set then $\mathrm{Aut}(E) = \mathrm{Sym}(E)$.
2. If E is a linearly ordered set then $\mathrm{Aut}(E)$ is isomorphic to the trivial group.

It is interesting that the automorphism group is the full symmetric group if and only if the order is the discrete order, but there are non-linear orders with trivial automorphism group.

The importance of automorphism groups of ordered sets for our purpose can be seen in the next observation. Let $(S, \preceq)$ be an ordered set. Then two ordered semigroups $(S, *, \preceq)$, $(S, \circ, \preceq)$ are isomorphic if and only if there is $\pi \in \mathrm{Aut}((S, \preceq))$ such that $\pi(x * y) = \pi(x) \circ \pi(y)$ for every $x, y \in S$.

Example 2.1 We give a list of all non-isomorphic ordered sets E with $\mathrm{card}(E) = 3$. We present also the corresponding automorphism groups $\mathrm{Aut}(E)$. A permutation $\pi \in \mathrm{Sym}(3)$ is written in the form $\pi(1)\pi(2)\pi(3)$. Remember that $G(3) = 5$.

i	E_i	$\mathrm{Aut}(E_i)$
1	00.0	123, 132, 213, 231, 312, 321
2	10.0	123
3	11.0	123, 132
4	01.1	123, 213
5	11.1	123

Let n be a positive integer. Denote by $\mathcal{OSET}(n)$ a set of representatives of all isomorphism classes of ordered sets with elements $1, 2, \ldots, n$.

In the next section we describe a construction of all non-isomorphic ordered semigroups of size at most 7. For $n \in \{1, 2, 3, 4, 5, 6, 7\}$, we need to know $\mathcal{OSET}(n)$ and automorphism groups for all members of $\mathcal{OSET}(n)$.

Let A be a binary matrix of size $n \times n$. We define a binary relation ρ_A on the set $\{1, 2, \ldots, n\}$ in the following way: For $i, j \in \{1, 2, \ldots, n\}$, $i \rho_A j$ if and only if $a_{ij} = 1$.

We have determined a set $\mathcal{OSET}(n)$ in the following way:

By brute-force computations we have found binary matrices $A_1, A_2, \ldots, A_l$ of size $n \times n$ with 0 under the main diagonal and with 1 on the main diagonal such that

$$(\{1, 2, \ldots, n\}, \rho_{A_1}), \ldots, (\{1, 2, \ldots, n\}, \rho_{A_l})$$

are all non-isomorphic ordered sets with n elements. We have $\mathcal{OSET}(n) = \{A_1, A_2, \ldots, A_l\}$. Of course, $l = G(n)$.

Our construction of $\mathcal{OSET}(n)$ is not efficient, but brute-force. An efficient method to construct pairwise non-isomorphic ordered sets was described by Brinkmann and McKay in [2].

Let $A \in \mathcal{OSET}(n)$. We determine the group $\mathrm{Aut}(A)$. We check, for every permutation $\pi \in \mathrm{Sym}(n)$, whether $i \rho_A j$ and $\pi(i) \rho_A \pi(j)$ are equivalent for every $i, j \in \{1, 2, \ldots, n\}$. Permutations, satisfying the condition, are members of $\mathrm{Aut}(A)$.

Again, the method is not efficient, but brute-force. Note that Pfeiffer in [10] enumerated the ordered sets on up to 12 points and determined their automorphism groups (see also [17], Sequence A091070).

3 All ordered semigroups of size at most 7

Let n be a positive integer. The set $\{1, 2, \ldots, n\}$ will be denoted by $[n]$.

Recall that $G(n)$ denotes the number of all non-isomorphic ordered sets of size n.

We construct all non-isomorphic ordered semigroups of size n in two steps. At first, we determine a set of representatives of all isomorphism classes of ordered sets with n elements

$$\mathcal{OSET}(n) = \{([n], \preceq_1), ([n], \preceq_2), \ldots, ([n], \preceq_{G(n)})\}.$$

The construction of the set $\mathcal{OSET}(n)$ is described in Sect. 2. Then, for every $i \in \{1, \ldots, G(n)\}$, we find all non-isomorphic ordered semigroups of the form $([n], *, \preceq_i)$. Let us denote them $([n], *_{i1}, \preceq_i)$, ..., $([n], *_{ip_i}, \preceq_i)$. Then

$$([n], *_{ij}, \preceq_i) \quad i = 1, \ldots, G(n), \;\; j = 1, \ldots, p_i$$

are all non-isomorphic ordered semigroups with n elements.

3.1 Ordered 2-nilpotent semigroups

Let S be a semigroup, $A \subseteq S$, $B \subseteq S$. We put $AB = \{xy |\, x \in A,\;\, y \in B\}$. Further, for positive integers n we define sets S^n recursively in the following way:

1. $S^1 = S$
2. $S^n = SS^{n-1}$ for $n > 1$.

Let k be a positive integer. Recall that by a *k-nilpotent semigroup* we mean a semigroup S with the properties $\mathrm{card}(S^k) = 1$ and $\mathrm{card}(S^l) > 1$ for every integer l with $0 < l < k$. Thus 1-nilpotent semigroups are exactly semigroups with one element, and a semigroup S is 2-nilpotent if and only if $\mathrm{card}(S) > 1$ and there exists $b \in S$ such that $xy = b$ for all $x, y \in S$.

Let G be a group. We recall the notion of G-sets. Let X be a set, α be a function (called an *action*), $\alpha : G \times X \to X, \alpha : (g, x) \mapsto gx$. Then X is a *G-set* if

1. $1x = x$ for all $x \in X$
2. $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$.

One also says that G *acts* on X. The relation $\equiv$ on X, defined by $x \equiv y$ if and only if $y = gx$ for some $g \in G$ $(x, y \in X)$, is an equivalence relation whose equivalence classes are called *G-orbits* (or simply *orbits*).

Theorem 3.1 *If a finite group G acts on a finite set X and N is the number of G-orbits of X, then*

$$N = \frac{1}{\mathrm{card}(G)} \cdot \sum_{\tau \in G} F(\tau),$$

where $F(\tau)$ is the number of $x \in X$ fixed by $\tau \in G$.

Proof e.g. [13], Theorem 3.22. q.e.d.

Let n be an integer, $n \geq 2$. We define, for any $b \in [n]$, a binary operation $\cdot_b$ on $[n]$ by the rule

$$x \cdot_b y = b$$

for all $x, y \in [n]$. Clearly, $([n], \cdot_b)$ is a 2-nilpotent semigroup. Let $b, c \in [n]$. Then $([n], \cdot_b) \cong ([n], \cdot_c)$. There is, up to isomorphism, only one 2-nilpotent semigroup with n elements.

Let $([n], \preceq)$ be an ordered set. Let $x, y, z \in [n]$. If $x \preceq y$ then $x \cdot_b z \preceq y \cdot_b z$ and $z \cdot_b x \preceq z \cdot_b y$, because all four products $x \cdot_b z, y \cdot_b z, z \cdot_b x, z \cdot_b y$ equal b. Hence $([n], \cdot_b, \preceq)$ is an ordered 2-nilpotent semigroup for every $b \in [n]$.

Lemma 3.2 *Let $b, c \in [n]$. Then $([n], \cdot_b, \preceq) \cong ([n], \cdot_c, \preceq)$ if and only if $\pi(b) = c$ for some $\pi \in \mathrm{Aut}(([n], \preceq))$.*

Proof 1. Suppose that $([n], \cdot_b, \preceq) \cong ([n], \cdot_c, \preceq)$. So, there is $\pi \in \mathrm{Aut}(([n], \preceq))$, $\pi(x \cdot_b y) = \pi(x) \cdot_c \pi(y)$ for any $x, y \in [n]$. Consequently, there is $\pi \in \mathrm{Aut}(([n], \preceq))$, $\pi(b) = c$.
2. Suppose that there exists $\pi \in \mathrm{Aut}(([n], \preceq))$, $\pi(b) = c$. Let $x, y \in [n]$. Then $\pi(x \cdot_b y) = \pi(b) = c = \pi(x) \cdot_c \pi(y)$. So, $([n], \cdot_b, \preceq) \cong ([n], \cdot_c, \preceq)$.
q.e.d.

Ordered 2-nilpotent semigroups lead to examples in which the semigroups are isomorphic and the ordered sets are isomorphic but the ordered semigroups are not.

The group $\mathrm{Aut}(([n], \preceq))$ acts on $[n]$ by $(\pi, x) \mapsto \pi(x)$ (for $\pi \in \mathrm{Aut}(([n], \preceq))$, $x \in [n]$). It follows from Lemma 3.2 that $([n], \cdot_b, \preceq) \cong ([n], \cdot_c, \preceq)$ if and only if $b \equiv c$.

Thus the number of all non-isomorphic ordered 2-nilpotent semigroups on $[n]$ with the order relation $\preceq$ is equal to the number of $\mathrm{Aut}(([n], \preceq))$-orbits.

Denote by $O2NS(n)$ the number of all non-isomorphic ordered 2-nilpotent semigroups of size n. We use Theorem 3.1 and obtain

$$O2NS(n) = \sum_{E \in \mathcal{OSET}(n)} \left(\frac{1}{\mathrm{card}(\mathrm{Aut}(E))} \cdot \sum_{\tau \in \mathrm{Aut}(E)} F(\tau) \right),$$

where $F(\tau)$ is the number of fixed points of the permutation τ.

Example 3.3 We use the formula for $O2NS(n)$ and Example 2.1. If $n = 3$, we have

$$\begin{aligned} O2NS(3) &= \sum_{i=1}^{5} \left(\frac{1}{\mathrm{card}(\mathrm{Aut}(E_i))} \cdot \sum_{\tau \in \mathrm{Aut}(E_i)} F(\tau) \right) \\ &= \frac{1}{6}(3+1+1+0+0+1) + \frac{1}{1}3 + \frac{1}{2}(3+1) + \frac{1}{2}(3+1) + \frac{1}{1}3 \\ &= 11. \end{aligned}$$

Using the formula for $O2NS(n)$, the values $O2NS(n)$ have been calculated for every $n \leq 12$. The values are presented in Table 2. We were able to compute the values

Table 2 Numbers of non-isomorphic ordered 2-nilpotent semigroups

n	2	3	4	5	6	7	8	9	10	11	12
$O2NS(n)$	3	11	47	243	1533	12038	118818	1487301	23738557	484673601	12677658783

$O2NS(n)$ for $n \leq 7$ and the values $O2NS(n)$ for $n \leq 12$ were computed by Goetz Pfeiffer from his data on automorphism groups of partial orders (see [10]).

This example shows the difference between semigroups and ordered semigroups: there is only one 2-nilpotent semigroup with 7 elements but there are 12038 ordered 2-nilpotent semigroups with 7 elements (of course, the number 7 could be replaced here).

3.2 The algorithm

We used computer for construction of all non-isomorphic ordered semigroups of size $n \leq 7$.

Let n be a positive integer. Let $([n], \preceq)$ be an ordered set. We want to construct all non-isomorphic ordered semigroups of the form $([n], *, \preceq)$.

Let $M_n(U)$ be the set of all $n \times n$ matrices over a set U. Any $A = (a_{ij}) \in M_n([n])$ determines a binary operation $*_A$ on $[n]$ by the rule

$$i *_A j = a_{ij}$$

$(i, j \in [n])$. So, A is the Cayley table of the operation $*_A$.

For each permutation $\pi \in \mathrm{Sym}(n)$ and each matrix $A \in M_n([n])$ we define

$$\pi(A) = B = (b_{ij}) \in M_n([n])$$

where

$$b_{ij} = \pi\left(a_{\pi^{-1}(i),\pi^{-1}(j)}\right)$$

(for $i, j \in [n]$).

It holds, for every $\pi, \rho \in \mathrm{Sym}(n)$, $A \in M_n([n])$,

$$\mathrm{id}(A) = A, \; (\rho\pi)(A) = \rho(\pi(A)).$$

Thus $\mathrm{Sym}(n)$ acts on $M_n([n])$ via $(\pi, A) \mapsto \pi(A)$.

Denote by $C_n([n], \preceq)$ the set of all $A \in M_n([n])$ such that the operation $*_A$ is associative and *compatible* with the relation $\preceq$ (i.e. $i \preceq j$ implies $i *_A k \preceq j *_A k$ and $k *_A i \preceq k *_A j$, for every $i, j, k \in [n]$). Thus,

$$C_n([n], \preceq) = \{A \in M_n([n]) | \; ([n], *_A, \preceq) \text{ is an ordered semigroup}\}.$$

We define a binary relation $\equiv$ on $C_n([n], \preceq)$ in the following way: For $A, B \in C_n([n], \preceq)$, $A \equiv B$ if and only if $\pi(A) = B$ for some $\pi \in \mathrm{Aut}(([n], \preceq))$.

Lemma 3.4 *The relation $\equiv$ is an equivalence relation on $C_n([n], \preceq)$ and, for every $A, B \in C_n([n], \preceq)$, $([n], *_A, \preceq) \cong ([n], *_B, \preceq)$ if and only if $A \equiv B$.*

Proof 1. $\mathrm{Sym}(n)$ acts on $M_n([n])$, $C_n([n], \preceq) \subseteq M_n([n])$ and $\mathrm{Aut}(([n], \preceq))$ is a subgroup in $\mathrm{Sym}(n)$. Consequently, the relation $\equiv$ is an equivalence relation on $C_n([n], \preceq)$.

2. Let $A, B \in C_n([n], \preceq)$. Let $([n], *_A, \preceq) \cong ([n], *_B, \preceq)$. There exists $\pi \in \mathrm{Aut}(([n], \preceq))$ such that $\pi(k *_A l) = \pi(k) *_B \pi(l)$ for every $k, l \in [n]$. Let $i, j \in [n]$. Then

$$\begin{aligned} b_{ij} &= i *_B j \\ &= \pi(\pi^{-1}(i)) *_B \pi(\pi^{-1}(j)) \\ &= \pi(\pi^{-1}(i) *_A \pi^{-1}(j)) \\ &= \pi(a_{\pi^{-1}(i), \pi^{-1}(j)}). \end{aligned}$$

Consequently, $\pi(A) = B$, $A \equiv B$.

3. Let $A, B \in C_n([n], \preceq)$. Let $A \equiv B$. There exists $\pi \in \mathrm{Aut}(([n], \preceq))$ such that $\pi(A) = B$. Let $i, j \in [n]$. Then

$$\begin{aligned} \pi(i *_A j) &= \pi(a_{ij}) \\ &= \pi(a_{\pi^{-1}(\pi(i)), \pi^{-1}(\pi(j))}) \\ &= b_{\pi(i), \pi(j)} \\ &= \pi(i) *_B \pi(j). \end{aligned}$$

Consequently, $\pi : ([n], *_A, \preceq) \to ([n], *_B, \preceq)$ is an isomorphism.
q.e.d.

Thus, we are interested in the factor set $C_n([n], \preceq)/_{\equiv}$.

Let $C_n([n], \preceq)/_{\equiv} = \{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_p\}$. Note that p is the number of all non-isomorphic ordered semigroups of the form $([n], *, \preceq)$.

The set $M_n([n])$ is linearly ordered by the relation $\leq$ defined by the following rule: Let $A, B \in M_n([n])$. Then

$$A \leq B$$

if $a_{ij} = b_{ij}$ for each $i, j \in [n]$ or there is a pair $k, l \in [n]$ such that

$$a_{kl} < b_{kl}$$

(the symbol $<$ denotes here the strong usual order of integers) and $a_{ij} = b_{ij}$ for all $i, j \in [n]$ with

$$j + (i - 1) \cdot n < l + (k - 1) \cdot n.$$

As usual, $A < B$ means $A \leq B$ and $A \neq B$.

We will construct matrices $A_1, \ldots, A_p$ which satisfy, for any $i \in \{1, \ldots, p\}$,

1. $A_i \in \mathcal{T}_i$
2. $A_i \leq B$ for all $B \in \mathcal{T}_i$.

Let $A = (a_{ij}) \in M_n([n] \cup \{0\})$. We can think of the matrix A as a table of a partial binary operation $*_A$ on $[n]$. Let $i, j \in [n]$. If $a_{ij} = 0$ then $i *_A j$ is undefined. If $a_{ij} \neq 0$ then $i *_A j = a_{ij}$.

Let $A = (a_{ij}) \in M_n([n] \cup \{0\})$. We define a set $R(A)$ of matrices by

$$R(A) = \{B \in M_n([n]) | \text{ for all } i, j \in [n],\ a_{ij} \neq 0 \text{ implies } b_{ij} = a_{ij}\}.$$

Thus $R(A)$ is the set of all matrices which are obtained if all zeros in A are replaced by elements of $[n]$.

Again, let $A = (a_{ij}) \in M_n([n] \cup \{0\})$. Let $k \in [n]$. We define a matrix $k(A) = B = (b_{ij}) \in M_n([n])$ by

$$b_{ij} = \begin{cases} a_{ij} & a_{ij} \neq 0 \\ k & a_{ij} = 0 \end{cases}$$

Thus $k(A)$ is the matrix which is obtained if all zeros in A are replaced by k. Obviously, $k(A) \in R(A)$. Further, if $A \in M_n([n])$ then $k(A) = A$.

It is clear that

$$1(A) \leq B \leq n(A)$$

for all $B \in R(A)$.

For each permutation $\pi \in \mathrm{Sym}(n)$ and each matrix $A \in M_n([n] \cup \{0\})$ we define

$$\overline{\pi}(A) = B = (b_{ij}) \in M_n([n] \cup \{0\})$$

where

$$b_{ij} = \begin{cases} \pi(a_{\pi^{-1}(i),\pi^{-1}(j)}) & a_{\pi^{-1}(i),\pi^{-1}(j)} \neq 0 \\ 0 & a_{\pi^{-1}(i),\pi^{-1}(j)} = 0 \end{cases}$$

Obviously, if $A \in M_n([n])$ then $\overline{\pi}(A) = \pi(A)$.

Our algorithm is based on the ideas presented by Plemmons in [12] (Plemmons was interested in the construction of non-equivalent finite semigroups). The difference lies in the fact that we use the automorphism group $\mathrm{Aut}(([n], \preceq))$ instead of the symmetric group $\mathrm{Sym}(n)$.

Algorithm 3.5 Input: An ordered set $([n], \preceq)$ for some positive integer n.

Output: $A_1, \ldots, A_p \in M_n([n])$ such that $([n], *_{A_1}, \preceq), \ldots, ([n], *_{A_p}, \preceq)$ are all non-isomorphic ordered semigroups of the form $([n], *, \preceq)$.

The algorithm works with a matrix $A = (a_{kl}) \in M_n([n] \cup \{0\})$ and with integers q, i, j, $1 \leq i \leq n$, $1 \leq j \leq n$.

1. (Initiation.) $a_{kl} \leftarrow 1$ for all $k, l \in [n]$, $q \leftarrow 1$, $i \leftarrow n$, $j \leftarrow n$, $A_q \leftarrow A$.
2. If $a_{ij} < n$ then $a_{ij} \leftarrow a_{ij} + 1$ and go to step 4.
3. (We have $a_{ij} = n$.)
   (i) $a_{ij} \leftarrow 0$
   (ii) If $j > 1$ then $j \leftarrow j - 1$ and go to step 2.
   (iii) (We have $j = 1$.) If $i > 1$ then $i \leftarrow i - 1$, $j \leftarrow n$ and go to step 2.
   (iv) (We have $j = i = 1$.) The process is complete and we have all the matrices $A_1, A_2, \dots, A_p$.
4. (Test for associativity.) If $a_{lm} \neq 0$, $a_{k,a_{lm}} \neq 0$, $a_{kl} \neq 0$, $a_{a_{kl},m} \neq 0$ and $a_{k,a_{lm}} \neq a_{a_{kl},m}$ for some $k, l, m \in [n]$ then go to step 2.
5. (Test for compatibility.) If $k \preceq l$, $a_{mk} \neq 0$, $a_{ml} \neq 0$, $a_{mk} \not\preceq a_{ml}$ or $k \preceq l$, $a_{km} \neq 0$, $a_{lm} \neq 0$, $a_{km} \not\preceq a_{lm}$ for some $k, l, m \in [n]$ then go to step 2.
6. (Test for isomorphism.) If $n(\overline{\pi}(A)) < 1(A)$ for some $\pi \in \mathrm{Aut}(([n], \preceq))$ then go to step 2.
7. (i) If $j < n$ then $j \leftarrow j + 1$ and go to step 2.
   (ii) (We have $j = n$.) If $i < n$ then $i \leftarrow i + 1$, $j \leftarrow 1$ and go to step 2.
   (iii) (We have $j = i = n$.) $q \leftarrow q + 1$, $A_q \leftarrow A$ and go to step 2.

Now, in order to construct all non-isomorphic ordered semigroups with n elements, we have to find all non-isomorphic ordered sets $([n], \preceq_i)$ with n elements, $i = 1, 2, \dots, G(n)$ (recall that $G(n)$ denotes the number of all non-isomorphic ordered sets with n elements), and to apply Algorithm 3.5 to every $([n], \preceq_i)$, $i = 1, 2, \dots, G(n)$. This two step process is described at the beginning of Sect. 3. Of course, obtained ordered semigroups are sorted by the type of their orderings.

3.3 Results and classification

We are going to present results of our computations.

We used a computer with quad core 2.40 GHz CPU and 2 GB RAM. We used 3 cores in parallel.

We have constructed all non-isomorphic ordered semigroups with n elements where $n \leq 7$. The constructed ordered semigroups of size $n \leq 6$ are available in the Internet at http://physics.ujep.cz/CZ/!usr_files/download-rucne/

The constructed ordered semigroups of size 7 were not stored. In our article we present only the enumeration of ordered semigroups of size $n \leq 7$.

The computations lasted 30 days (75 min for ordered semigroups of size $n \leq 6$).

In Table 3 the numbers of ordered semigroups with n elements are given, step by step for $n = 1, 2, 3, 4, 5, 6$. The binary sequences in the first column of any table determine the order relation on $[n]$, in the way explained in Sect. 2. The symbols **s**, **c**, **m**, **b**, **r**, **i**, **2**, **3** in the heads of the tables are meant as abbreviations:

s—semigroup

c—commutative semigroup ($xy = yx$ for all $x, y \in S$)

m—monoid (S contains an identity)

b—band ($x^2 = x$ for all $x \in S$)

r—regular semigroup (for all $x \in S$ there exists an element $y \in S$ such that $xyx = x$ and $yxy = y$)

Table 3 Numbers of non-isomorphic ordered semigroups

	s	**c**	**m**	**b**	**r**	**i**	**2**	**3**
1 Element								
	1	1	1	1	1	1	0	0
Sum	1	1	1	1	1	1	0	0
2 Elements								
0	5	3	2	3	4	2	1	0
1	6	4	2	4	4	2	2	0
Sum	11	7	4	7	8	4	3	0
3 Elements								
0 0. 0	24	12	7	10	13	5	1	1
1 0. 0	53	27	10	20	22	8	3	2
1 1. 0	26	12	6	11	12	4	2	1
0 1. 1	26	12	6	11	12	4	2	1
1 1. 1	44	20	8	17	17	5	3	2
Sum	173	83	37	69	76	26	11	7
4 Elements								
0 0 0. 0 0. 0	188	58	35	46	67	16	1	9
1 0 0. 0 0. 0	591	181	73	136	158	34	3	23
1 1 0. 0 0. 0	324	101	36	70	81	19	3	17
1 1 1. 0 0. 0	178	51	28	47	56	12	2	9
0 1 0. 1 0. 0	324	101	36	70	81	19	3	17
1 1 0. 1 0. 0	584	176	58	119	128	26	4	34
0 0 1. 1 0. 0	210	64	28	66	70	12	2	10
0 1 1. 1 0. 0	210	82	12	40	42	10	4	14
1 1 1. 1 0. 0	439	133	56	90	92	18	4	27
0 1 1. 1 1. 0	48	20	0	2	2	0	2	2
1 1 1. 1 1. 0	220	69	19	41	45	9	3	12
0 0 1. 0 1. 1	178	51	28	47	56	12	2	9
1 0 1. 0 1. 1	439	133	56	90	92	18	4	27
1 1 1. 0 1. 1	214	65	31	48	52	11	3	10
0 1 1. 1 1. 1	220	69	19	41	45	9	3	12
1 1 1. 1 1. 1	386	114	34	82	82	14	4	14
Sum	4753	1468	549	1035	1149	239	47	246
5 elements								
0 0 0 0. 0 0 0. 0 0. 0	1915	325	228	251	355	52	1	118
1 0 0 0. 0 0 0. 0 0. 0	8123	1335	720	1010	1230	160	3	335
1 1 0 0. 0 0 0. 0 0. 0	5084	842	396	571	692	96	3	342
1 1 1 0. 0 0 0. 0 0. 0	3203	546	224	374	454	68	3	298
1 1 1 1. 0 0 0. 0 0. 0	1740	277	199	244	313	44	2	118
0 1 0 0. 1 0 0. 0 0. 0	5084	842	396	571	692	96	3	342
1 1 0 0. 1 0 0. 0 0. 0	9445	1509	680	1018	1159	139	4	677

Table 3 continued

	s	c	m	b	r	i	2	3
0 0 1 0. 1 0 0. 0 0. 0	5481	952	426	846	904	101	3	240
0 1 1 0. 1 0 0. 0 0. 0	4393	923	230	380	406	64	5	322
1 1 1 0. 1 0 0. 0 0. 0	8820	1540	531	807	859	115	5	876
0 0 1 1. 1 0 0. 0 0. 0	2556	473	220	430	465	55	4	171
0 1 1 1. 1 0 0. 0 0. 0	1750	415	46	214	253	37	4	154
1 1 1 1. 1 0 0. 0 0. 0	6135	1015	548	673	748	92	4	499
0 1 1 0. 1 1 0. 0 0. 0	965	219	48	20	20	0	3	46
1 1 1 0. 1 1 0. 0 0. 0	4398	789	254	378	426	58	4	383
0 1 0 1. 1 1 0. 0 0. 0	502	156	23	58	60	10	3	74
0 1 1 1. 1 1 0. 0 0. 0	745	228	0	12	12	0	4	65
1 1 1 1. 1 1 0. 0 0. 0	4039	719	366	322	352	46	4	410
0 1 1 1. 1 1 1. 0 0. 0	202	59	0	2	2	0	2	10
1 1 1 1. 1 1 1. 0 0. 0	1774	328	126	172	207	27	3	148
0 0 1 0. 0 1 0. 1 0. 0	3203	546	224	374	454	68	3	298
1 0 1 0. 0 1 0. 1 0. 0	8820	1540	531	807	859	115	5	876
1 1 1 0. 0 1 0. 1 0. 0	3950	714	266	383	434	62	4	269
0 0 0 1. 0 1 0. 1 0. 0	2556	473	220	430	465	55	4	171
0 0 1 1. 0 1 0. 1 0. 0	1750	415	46	214	253	37	4	154
1 0 1 1. 0 1 0. 1 0. 0	3132	686	80	289	295	41	5	315
1 1 1 1. 0 1 0. 1 0. 0	2986	564	218	306	334	48	4	208
0 1 1 0. 1 1 0. 1 0. 0	4398	789	254	378	426	58	4	383
1 1 1 0. 1 1 0. 1 0. 0	7637	1325	450	780	822	94	5	394
0 0 0 1. 1 1 0. 1 0. 0	4756	854	396	750	762	82	5	299
0 0 1 1. 1 1 0. 1 0. 0	1767	437	69	191	195	25	5	209
0 1 1 1. 1 1 0. 1 0. 0	2471	597	53	202	211	31	5	186
1 1 1 1. 1 1 0. 1 0. 0	5269	953	402	544	553	63	5	279
0 0 1 1. 0 0 1. 1 0. 0	502	156	23	58	60	10	3	74
1 0 1 1. 0 0 1. 1 0. 0	1767	437	69	191	195	25	5	209
1 1 1 1. 0 0 1. 1 0. 0	2304	408	211	278	284	31	3	174
0 0 1 1. 0 1 1. 1 0. 0	745	228	0	12	12	0	4	65
1 0 1 1. 0 1 1. 1 0. 0	2471	597	53	202	211	31	5	186
0 1 1 1. 0 1 1. 1 0. 0	1122	334	0	8	8	0	5	121
1 1 1 1. 0 1 1. 1 0. 0	3203	671	209	182	190	30	5	255
0 1 1 1. 1 1 1. 1 0. 0	826	230	0	4	4	0	4	53
1 1 1 1. 1 1 1. 1 0. 0	4308	838	268	358	366	44	5	232
0 0 1 1. 0 1 1. 1 1. 0	202	59	0	2	2	0	2	10
1 0 1 1. 0 1 1. 1 1. 0	826	230	0	4	4	0	4	53
1 1 1 1. 0 1 1. 1 1. 0	792	170	42	10	11	3	3	49
0 1 1 1. 1 1 1. 1 1. 0	876	200	0	106	129	17	3	23

Table 3 continued

	s	c	m	b	r	i	2	3
1 1 1 1. 1 1 1. 1 1. 0	2120	417	96	185	202	24	4	82
0 0 0 1. 0 0 1. 0 1. 1	1740	277	199	244	313	44	2	118
1 0 0 1. 0 0 1. 0 1. 1	6135	1015	548	673	748	92	4	499
1 1 0 1. 0 0 1. 0 1. 1	2986	564	218	306	334	48	4	208
1 1 1 1. 0 0 1. 0 1. 1	1206	239	115	182	217	30	3	54
0 1 0 1. 1 0 1. 0 1. 1	4039	719	366	322	352	46	4	410
1 1 0 1. 1 0 1. 0 1. 1	5269	953	402	544	553	63	5	279
0 0 1 1. 1 0 1. 0 1. 1	2304	408	211	278	284	31	3	174
0 1 1 1. 1 0 1. 0 1. 1	3203	671	209	182	190	30	5	255
1 1 1 1. 1 0 1. 0 1. 1	3855	761	286	406	414	50	5	170
0 1 1 1. 1 1 1. 0 1. 1	792	170	42	10	11	3	3	49
1 1 1 1. 1 1 1. 0 1. 1	2071	409	130	210	228	29	4	67
0 0 1 1. 0 1 1. 1 1. 1	1774	328	126	172	207	27	3	148
1 0 1 1. 0 1 1. 1 1. 1	4308	838	268	358	366	44	5	232
1 1 1 1. 0 1 1. 1 1. 1	2071	409	130	210	228	29	4	67
0 1 1 1. 1 1 1. 1 1. 1	2120	417	96	185	202	24	4	82
1 1 1 1. 1 1 1. 1 1. 1	3852	710	184	422	422	42	5	83
6 elements								
0 0 0 0 0. 0 0 0 0. 0 0 0. 0 0. 0	28634	2143	2237	1682	2471	208	1	4671
1 0 0 0 0. 0 0 0 0. 0 0 0. 0 0. 0	144825	10894	9335	8624	10854	814	3	16550
1 1 0 0 0. 0 0 0 0. 0 0 0. 0 0. 0	103461	7566	5674	4987	6242	522	3	17744
1 1 1 0 0. 0 0 0 0. 0 0 0. 0 0. 0	82570	5710	3633	3592	4550	406	3	21916
1 1 1 1 0. 0 0 0 0. 0 0 0. 0 0. 0	54385	3886	2066	2408	3049	294	3	16197
1 1 1 1 1. 0 0 0 0. 0 0 0. 0 0. 0	24823	1815	1863	1506	2017	174	2	4671
0 1 0 0 0. 1 0 0 0. 0 0 0. 0 0. 0	103461	7566	5674	4987	6242	522	3	17744
1 1 0 0 0. 1 0 0 0. 0 0 0. 0 0. 0	197107	13854	10331	9271	10997	794	4	35407
0 0 1 0 0. 1 0 0 0. 0 0 0. 0 0. 0	143438	11954	7833	10738	12046	795	3	11092
0 1 1 0 0. 1 0 0 0. 0 0 0. 0 0. 0	108901	10111	4679	4034	4558	416	5	15853
1 1 1 0 0. 1 0 0 0. 0 0 0. 0 0. 0	238652	17038	9611	8245	9293	743	5	63552
0 0 1 1 0. 1 0 0 0. 0 0 0. 0 0. 0	97382	8645	4591	7033	7802	578	5	11979
0 1 1 1 0. 1 0 0 0. 0 0 0. 0 0. 0	58648	6064	1826	2526	2927	287	5	10604
1 1 1 1 0. 1 0 0 0. 0 0 0. 0 0. 0	219822	15670	7031	7443	8357	709	5	71595
0 0 1 1 1. 1 0 0 0. 0 0 0. 0 0. 0	28490	2546	1558	2356	2742	214	4	4788
0 1 1 1 1. 1 0 0 0. 0 0 0. 0 0. 0	23667	2811	289	1305	1602	152	4	4773
1 1 1 1 1. 1 0 0 0. 0 0 0. 0 0. 0	124342	9076	7242	5575	6480	508	4	28910
0 1 1 0 0. 1 1 0 0. 0 0 0. 0 0. 0	23620	2418	1035	254	282	0	3	2743
1 1 1 0 0. 1 1 0 0. 0 0 0. 0 0. 0	119842	8788	4801	3986	4723	379	4	29640
0 1 0 1 0. 1 1 0 0. 0 0 0. 0 0. 0	18055	2366	538	662	691	75	4	4184
0 1 1 1 0. 1 1 0 0. 0 0 0. 0 0. 0	26593	3428	745	206	206	0	5	3468
1 1 1 1 0. 1 1 0 0. 0 0 0. 0 0. 0	164730	11580	4643	3511	3899	349	5	64459

Table 3 continued

	s	c	m	b	r	i	2	3
0 0 0 1 1. 1 1 0 0. 0 0 0. 0 0. 0	10708	1016	821	968	1119	91	2	1406
0 1 0 1 1. 1 1 0 0. 0 0 0. 0 0. 0	11072	1712	237	567	654	70	5	3862
0 1 1 1 1. 1 1 0 0. 0 0 0. 0 0. 0	12285	1885	0	102	114	0	4	1998
1 1 1 1 1. 1 1 0 0. 0 0 0. 0 0. 0	115789	8262	5957	2980	3498	286	4	37887
0 1 1 1 0. 1 1 1 0. 0 0 0. 0 0. 0	6377	831	202	28	28	0	3	461
1 1 1 1 0. 1 1 1 0. 0 0 0. 0 0. 0	66233	5023	2000	1986	2387	209	4	21663
0 1 1 0 1. 1 1 1 0. 0 0 0. 0 0. 0	3757	759	0	46	46	0	3	1232
0 1 1 1 1. 1 1 1 0. 0 0 0. 0 0. 0	7607	1275	0	18	18	0	4	865
1 1 1 1 1. 1 1 1 0. 0 0 0. 0 0. 0	73901	5490	3856	1686	1976	170	4	25002
0 1 1 1 1. 1 1 1 1. 0 0 0. 0 0. 0	1768	285	0	2	2	0	2	119
1 1 1 1 1. 1 1 1 1. 0 0 0. 0 0. 0	24600	2035	1204	869	1125	97	3	6500
0 0 1 0 0. 0 1 0 0. 1 0 0. 0 0. 0	82570	5710	3633	3592	4550	406	3	21916
1 0 1 0 0. 0 1 0 0. 1 0 0. 0 0. 0	238652	17038	9611	8245	9293	743	5	63552
1 1 1 0 0. 0 1 0 0. 1 0 0. 0 0. 0	94417	7603	4413	3649	4326	366	4	17812
0 0 0 1 0. 0 1 0 0. 1 0 0. 0 0. 0	97382	8645	4591	7033	7802	578	5	11979
0 0 1 1 0. 0 1 0 0. 1 0 0. 0 0. 0	58648	6064	1826	2526	2927	287	5	10604
1 0 1 1 0. 0 1 0 0. 1 0 0. 0 0. 0	110881	10563	3264	3358	3511	339	6	22649
1 1 1 1 0. 0 1 0 0. 1 0 0. 0 0. 0	96631	8454	3351	3188	3557	351	5	21911
0 0 0 1 1. 0 1 0 0. 1 0 0. 0 0. 0	15140	1620	446	1362	1546	112	4	2650
0 0 1 1 1. 0 1 0 0. 1 0 0. 0 0. 0	12694	1740	168	1040	1410	132	4	2578
1 0 1 1 1. 0 1 0 0. 1 0 0. 0 0. 0	40182	4561	397	1797	2041	185	5	8973
1 1 1 1 1. 0 1 0 0. 1 0 0. 0 0. 0	61230	5421	3100	2587	3028	283	4	11329
0 1 1 0 0. 1 1 0 0. 1 0 0. 0 0. 0	119842	8788	4801	3986	4723	379	4	29640
1 1 1 0 0. 1 1 0 0. 1 0 0. 0 0. 0	189503	14705	8253	8296	9256	628	5	30565
0 0 0 1 0. 1 1 0 0. 1 0 0. 0 0. 0	187132	16076	8586	12934	13610	892	6	23878
0 0 1 1 0. 1 1 0 0. 1 0 0. 0 0. 0	62636	6718	1881	2346	2441	205	6	14811
0 1 1 1 0. 1 1 0 0. 1 0 0. 0 0. 0	88248	9338	2557	2567	2712	262	6	13697
1 1 1 1 0. 1 1 0 0. 1 0 0. 0 0. 0	172716	14930	5933	6175	6488	508	6	34708
0 0 0 1 1. 1 1 0 0. 1 0 0. 0 0. 0	32914	3269	1842	2820	3010	204	5	4260
0 0 1 1 1. 1 1 0 0. 1 0 0. 0 0. 0	13391	1879	268	961	1120	90	5	3171
0 1 1 1 1. 1 1 0 0. 1 0 0. 0 0. 0	25687	3505	213	1286	1495	135	5	3158
1 1 1 1 1. 1 1 0 0. 1 0 0. 0 0. 0	104740	8844	5437	4770	5255	377	5	16649
0 0 1 1 0. 0 0 1 0. 1 0 0. 0 0. 0	18055	2366	538	662	691	75	4	4184
1 0 1 1 0. 0 0 1 0. 1 0 0. 0 0. 0	62636	6718	1881	2346	2441	205	6	14811
1 1 1 1 0. 0 0 1 0. 1 0 0. 0 0. 0	74999	6282	2652	3085	3259	239	4	20179
0 0 0 0 1. 0 0 1 0. 1 0 0. 0 0. 0	19784	1844	1308	2220	2320	132	2	1376
0 0 1 0 1. 0 0 1 0. 1 0 0. 0 0. 0	36464	4246	1094	2422	2518	206	6	4912
0 1 1 0 1. 0 0 1 0. 1 0 0. 0 0. 0	81293	7521	4030	5258	5390	390	6	14105
0 0 1 1 1. 0 0 1 0. 1 0 0. 0 0. 0	7567	1287	124	427	475	53	4	1684
1 0 1 1 1. 0 0 1 0. 1 0 0. 0 0. 0	51130	5968	729	2148	2292	196	6	13268

Table 3 continued

	s	c	m	b	r	i	2	3
1 1 1 1 1. 0 0 1 0. 1 0 0. 0 0. 0	96935	7785	5354	4330	4547	325	4	21048
0 0 1 1 0. 0 1 1 0. 1 0 0. 0 0. 0	26593	3428	745	206	206	0	5	3468
1 0 1 1 0. 0 1 1 0. 1 0 0. 0 0. 0	88248	9338	2557	2567	2712	262	6	13697
0 1 1 1 0. 0 1 1 0. 1 0 0. 0 0. 0	39538	5030	1122	182	182	0	6	6525
1 1 1 1 0. 0 1 1 0. 1 0 0. 0 0. 0	115128	10464	3563	2013	2155	223	6	32487
0 0 1 0 1. 0 1 1 0. 1 0 0. 0 0. 0	7567	1287	124	427	475	53	4	1684
0 1 1 0 1. 0 1 1 0. 1 0 0. 0 0. 0	17333	2537	365	674	685	69	6	5178
0 0 0 1 1. 0 1 1 0. 1 0 0. 0 0. 0	5180	1062	46	208	212	26	6	2198
0 0 1 1 1. 0 1 1 0. 1 0 0. 0 0. 0	9128	1916	0	100	100	0	6	2080
1 0 1 1 1. 0 1 1 0. 1 0 0. 0 0. 0	46818	5706	322	1175	1221	121	6	12069
0 1 1 1 1. 0 1 1 0. 1 0 0. 0 0. 0	19867	3053	0	82	82	0	6	4489
1 1 1 1 1. 0 1 1 0. 1 0 0. 0 0. 0	116911	10335	4946	1871	1958	214	6	33272
0 1 1 1 0. 1 1 1 0. 1 0 0. 0 0. 0	28795	3522	826	118	118	0	5	2794
1 1 1 1 0. 1 1 1 0. 1 0 0. 0 0. 0	151025	13299	4784	4547	4785	377	6	31373
0 1 1 0 1. 1 1 1 0. 1 0 0. 0 0. 0	9537	1610	165	399	409	41	4	1722
0 0 0 1 1. 1 1 1 0. 1 0 0. 0 0. 0	11936	1818	287	510	516	50	6	4657
0 0 1 1 1. 1 1 1 0. 1 0 0. 0 0. 0	8641	1737	0	52	52	0	6	2734
0 1 1 1 1. 1 1 1 0. 1 0 0. 0 0. 0	20209	3501	0	74	74	0	6	2717
1 1 1 1 1. 1 1 1 0. 1 0 0. 0 0. 0	137864	12032	6911	3660	3759	299	6	31500
0 0 0 1 1. 0 0 1 1. 1 0 0. 0 0. 0	7274	1000	96	124	124	0	4	668
1 0 0 1 1. 0 0 1 1. 1 0 0. 0 0. 0	41251	3993	1778	2502	2706	198	5	6808
0 0 1 1 1. 0 0 1 1. 1 0 0. 0 0. 0	3088	740	0	24	24	0	5	627
1 0 1 1 1. 0 0 1 1. 1 0 0. 0 0. 0	20158	2419	250	634	697	59	5	6469
0 1 1 1 1. 0 0 1 1. 1 0 0. 0 0. 0	9750	1526	0	68	68	0	5	2192
1 1 1 1 1. 0 0 1 1. 1 0 0. 0 0. 0	63361	5315	3613	2117	2282	166	5	17491
0 0 1 1 1. 0 1 1 1. 1 0 0. 0 0. 0	2466	583	0	14	14	0	4	310
1 0 1 1 1. 0 1 1 1. 1 0 0. 0 0. 0	29454	3732	288	1031	1219	115	5	6429
0 1 1 1 1. 0 1 1 1. 1 0 0. 0 0. 0	8494	1503	0	10	10	0	5	1211
1 1 1 1 1. 0 1 1 1. 1 0 0. 0 0. 0	57740	5374	2535	909	1075	111	5	16527
0 1 1 1 1. 1 1 1 1. 1 0 0. 0 0. 0	7009	1185	0	4	4	0	4	621
1 1 1 1 1. 1 1 1 1. 1 0 0. 0 0. 0	80060	7344	3567	2530	2840	216	5	16514
0 0 1 1 0. 0 1 1 0. 1 1 0. 0 0. 0	6377	831	202	28	28	0	3	461
1 0 1 1 0. 0 1 1 0. 1 1 0. 0 0. 0	28795	3522	826	118	118	0	5	2794
1 1 1 1 0. 0 1 1 0. 1 1 0. 0 0. 0	25927	2504	872	120	133	17	4	6320
0 0 1 0 1. 0 1 1 0. 1 1 0. 0 0. 0	3088	740	0	24	24	0	5	627
0 0 1 1 1. 0 1 1 0. 1 1 0. 0 0. 0	2466	583	0	14	14	0	4	310
1 0 1 1 1. 0 1 1 0. 1 1 0. 0 0. 0	11522	1798	0	44	44	0	5	1860
1 1 1 1 1. 0 1 1 0. 1 1 0. 0 0. 0	25039	2296	1123	134	152	20	4	6376
0 1 1 1 0. 1 1 1 0. 1 1 0. 0 0. 0	28549	3072	876	1406	1670	148	4	1173
1 1 1 1 0. 1 1 1 0. 1 1 0. 0 0. 0	70885	6518	2302	2441	2701	207	5	12073

Table 3 continued

	s	c	m	b	r	i	2	3
0 0 1 0 1. 1 1 1 0. 1 1 0. 0 0. 0	14288	2348	242	443	455	51	6	3899
0 0 1 1 1. 1 1 1 0. 1 1 0. 0 0. 0	4822	1024	0	26	26	0	5	1044
0 1 1 1 1. 1 1 1 0. 1 1 0. 0 0. 0	16655	2605	140	808	910	88	5	1037
1 1 1 1 1. 1 1 1 0. 1 1 0. 0 0. 0	58664	5287	2858	1774	1924	150	5	11837
0 0 0 1 1. 0 1 0 1. 1 1 0. 0 0. 0	334	112	0	2	2	0	2	96
0 0 1 1 1. 0 1 0 1. 1 1 0. 0 0. 0	1778	522	0	2	2	0	4	240
1 0 1 1 1. 0 1 0 1. 1 1 0. 0 0. 0	14700	2436	0	18	18	0	6	3536
1 1 1 1 1. 0 1 0 1. 1 1 0. 0 0. 0	33954	3116	1569	313	321	41	4	12722
0 0 1 1 1. 0 1 1 1. 1 1 0. 0 0. 0	1292	374	0	2	2	0	4	116
1 0 1 1 1. 0 1 1 1. 1 1 0. 0 0. 0	12641	2051	0	24	24	0	5	1825
0 1 1 1 1. 0 1 1 1. 1 1 0. 0 0. 0	6261	1195	0	2	2	0	5	884
1 1 1 1 1. 0 1 1 1. 1 1 0. 0 0. 0	38737	3624	1780	86	97	15	5	11980
0 1 1 1 1. 1 1 1 1. 1 1 0. 0 0. 0	6139	1092	0	16	18	0	4	450
1 1 1 1 1. 1 1 1 1. 1 1 0. 0 0. 0	53135	5103	2280	1263	1378	112	5	11973
0 0 1 1 1. 0 1 1 1. 1 1 1. 0 0. 0	358	98	0	2	2	0	2	18
1 0 1 1 1. 0 1 1 1. 1 1 1. 0 0. 0	4061	727	0	4	4	0	4	323
1 1 1 1 1. 0 1 1 1. 1 1 1. 0 0. 0	8306	810	457	19	23	7	3	1926
0 1 1 1 1. 1 1 1 1. 1 1 1. 0 0. 0	5911	842	0	445	591	51	3	167
1 1 1 1 1. 1 1 1 1. 1 1 1. 0 0. 0	22159	2187	873	775	934	72	4	3800
0 0 0 1 0. 0 0 1 0. 0 1 0. 1 0. 0	54385	3886	2066	2408	3049	294	3	16197
1 0 0 1 0. 0 0 1 0. 0 1 0. 1 0. 0	219822	15670	7031	7443	8357	709	5	71595
1 1 0 1 0. 0 0 1 0. 0 1 0. 1 0. 0	96631	8454	3351	3188	3557	351	5	21911
1 1 1 1 0. 0 0 1 0. 0 1 0. 1 0. 0	31283	3184	1399	1688	2065	195	4	3177
0 0 0 0 1. 0 0 1 0. 0 1 0. 1 0. 0	28490	2546	1558	2356	2742	214	4	4788
0 0 0 1 1. 0 0 1 0. 0 1 0. 1 0. 0	23667	2811	289	1305	1602	152	4	4773
1 0 0 1 1. 0 0 1 0. 0 1 0. 1 0. 0	40182	4561	397	1797	2041	185	5	8973
1 1 0 1 1. 0 0 1 0. 0 1 0. 1 0. 0	35158	4201	360	1427	1537	158	5	6681
1 1 1 1 1. 0 0 1 0. 0 1 0. 1 0. 0	23983	2671	1081	1432	1669	159	4	2460
0 1 0 1 0. 1 0 1 0. 0 1 0. 1 0. 0	164730	11580	4643	3511	3899	349	5	64459
1 1 0 1 0. 1 0 1 0. 0 1 0. 1 0. 0	172716	14930	5933	6175	6488	508	6	34708
0 0 1 1 0. 1 0 1 0. 0 1 0. 1 0. 0	74999	6282	2652	3085	3259	239	4	20179
0 1 1 1 0. 1 0 1 0. 0 1 0. 1 0. 0	115128	10464	3563	2013	2155	223	6	32487
1 1 1 1 0. 1 0 1 0. 0 1 0. 1 0. 0	111399	11225	4339	4152	4420	368	6	11966
0 0 0 0 1. 1 0 1 0. 0 1 0. 1 0. 0	81293	7521	4030	5258	5390	390	6	14105
0 0 0 1 1. 1 0 1 0. 0 1 0. 1 0. 0	51130	5968	729	2148	2292	196	6	13268
0 1 0 1 1. 1 0 1 0. 0 1 0. 1 0. 0	46818	5706	322	1175	1221	121	6	12069
1 1 0 1 1. 1 0 1 0. 0 1 0. 1 0. 0	59594	7044	700	2290	2336	192	6	9958
0 0 1 1 1. 1 0 1 0. 0 1 0. 1 0. 0	30745	3805	617	1537	1561	121	6	6361
0 1 1 1 1. 1 0 1 0. 0 1 0. 1 0. 0	33269	4553	258	886	926	104	6	5725
1 1 1 1 1. 1 0 1 0. 0 1 0. 1 0. 0	79902	8714	3188	3278	3376	290	6	8499

Table 3 continued

	s	**c**	**m**	**b**	**r**	**i**	**2**	**3**
0 1 1 1 0. 1 1 1 0. 0 1 0. 1 0. 0	25927	2504	872	120	133	17	4	6320
1 1 1 1 0. 1 1 1 0. 0 1 0. 1 0. 0	58108	5876	2307	2249	2512	213	5	4644
0 0 0 0 1. 1 1 1 0. 0 1 0. 1 0. 0	34522	3541	1932	2406	2578	200	5	3459
0 0 0 1 1. 1 1 1 0. 0 1 0. 1 0. 0	13465	1889	367	650	710	70	5	2575
0 1 0 1 1. 1 1 1 0. 0 1 0. 1 0. 0	19122	2966	283	455	462	52	6	3486
0 1 1 1 1. 1 1 1 0. 0 1 0. 1 0. 0	11294	1819	60	60	62	10	5	1443
1 1 1 1 1. 1 1 1 0. 0 1 0. 1 0. 0	38713	4329	1850	1598	1734	146	5	3962
0 0 0 0 1. 0 0 0 1. 0 1 0. 1 0. 0	10708	1016	821	968	1119	91	2	1406
1 0 0 0 1. 0 0 0 1. 0 1 0. 1 0. 0	32914	3269	1842	2820	3010	204	5	4260
0 0 0 1 1. 0 0 0 1. 0 1 0. 1 0. 0	11072	1712	237	567	654	70	5	3862
1 0 0 1 1. 0 0 0 1. 0 1 0. 1 0. 0	13391	1879	268	961	1120	90	5	3171
0 1 0 1 1. 0 0 0 1. 0 1 0. 1 0. 0	17333	2537	365	674	685	69	6	5178
1 1 0 1 1. 0 0 0 1. 0 1 0. 1 0. 0	30745	3805	617	1537	1561	121	6	6361
0 1 1 1 1. 0 0 0 1. 0 1 0. 1 0. 0	13465	1889	367	650	710	70	5	2575
1 1 1 1 1. 0 0 0 1. 0 1 0. 1 0. 0	34661	3613	1849	1820	1941	160	5	4320
0 0 0 1 1. 0 0 1 1. 0 1 0. 1 0. 0	12285	1885	0	102	114	0	4	1998
1 0 0 1 1. 0 0 1 1. 0 1 0. 1 0. 0	25687	3505	213	1286	1495	135	5	3158
0 1 0 1 1. 0 0 1 1. 0 1 0. 1 0. 0	19867	3053	0	82	82	0	6	4489
1 1 0 1 1. 0 0 1 1. 0 1 0. 1 0. 0	33269	4553	258	886	926	104	6	5725
0 1 1 1 1. 0 0 1 1. 0 1 0. 1 0. 0	8277	1366	0	50	56	0	5	1707
1 1 1 1 1. 0 0 1 1. 0 1 0. 1 0. 0	26803	3271	887	824	970	101	5	3456
0 1 0 1 1. 1 0 1 1. 0 1 0. 1 0. 0	11522	1798	0	44	44	0	5	1860
1 1 0 1 1. 1 0 1 1. 0 1 0. 1 0. 0	44983	5743	441	1548	1582	136	6	6298
0 0 1 1 1. 1 0 1 1. 0 1 0. 1 0. 0	4892	860	0	18	18	0	4	1207
0 1 1 1 1. 1 0 1 1. 0 1 0. 1 0. 0	13897	2419	0	30	30	0	6	2000
1 1 1 1 1. 1 0 1 1. 0 1 0. 1 0. 0	50377	5963	1707	1218	1243	121	6	6823
0 1 1 1 1. 1 1 1 1. 0 1 0. 1 0. 0	4692	798	0	2	2	0	4	433
1 1 1 1 1. 1 1 1 1. 0 1 0. 1 0. 0	34695	4038	1441	1199	1311	116	5	4043
0 0 1 1 0. 0 1 1 0. 1 1 0. 1 0. 0	66233	5023	2000	1986	2387	209	4	21663
1 0 1 1 0. 0 1 1 0. 1 1 0. 1 0. 0	151025	13299	4784	4547	4785	377	6	31373
1 1 1 1 0. 0 1 1 0. 1 1 0. 1 0. 0	58108	5876	2307	2249	2512	213	5	4644
0 0 0 0 1. 0 1 1 0. 1 1 0. 1 0. 0	41251	3993	1778	2502	2706	198	5	6808
0 0 0 1 1. 0 1 1 0. 1 1 0. 1 0. 0	20158	2419	250	634	697	59	5	6469
0 0 1 1 1. 0 1 1 0. 1 1 0. 1 0. 0	29454	3732	288	1031	1219	115	5	6429
1 0 1 1 1. 0 1 1 0. 1 1 0. 1 0. 0	44983	5743	441	1548	1582	136	6	6298
1 1 1 1 1. 0 1 1 0. 1 1 0. 1 0. 0	40559	4404	1679	1676	1810	157	5	2512
0 1 1 1 0. 1 1 1 0. 1 1 0. 1 0. 0	70885	6518	2302	2441	2701	207	5	12073
1 1 1 1 0. 1 1 1 0. 1 1 0. 1 0. 0	111971	10707	4204	5418	5632	364	6	6433

Table 3 continued

	s	c	m	b	r	i	2	3
0 0 0 0 1. 1 1 1 0. 1 1 0. 1 0. 0	73112	7054	3990	5270	5330	336	6	7670
0 0 0 1 1. 1 1 1 0. 1 1 0. 1 0. 0	25709	3493	675	1071	1088	88	6	5307
0 0 1 1 1. 1 1 1 0. 1 1 0. 1 0. 0	26593	3807	471	983	991	81	6	4196
0 1 1 1 1. 1 1 1 0. 1 1 0. 1 0. 0	35121	4955	420	1263	1319	113	6	3575
1 1 1 1 1. 1 1 1 0. 1 1 0. 1 0. 0	73248	7610	3447	3603	3645	241	6	4313
1 0 0 0 1. 0 0 0 1. 1 1 0. 1 0. 0	27572	2708	1430	2477	2498	154	3	1762
0 0 0 1 1. 0 0 0 1. 1 1 0. 1 0. 0	11936	1818	287	510	516	50	6	4657
1 0 0 1 1. 0 0 0 1. 1 1 0. 1 0. 0	12364	1858	368	888	888	62	6	2468
0 0 1 1 1. 0 0 0 1. 1 1 0. 1 0. 0	14288	2348	242	443	455	51	6	3899
1 0 1 1 1. 0 0 0 1. 1 1 0. 1 0. 0	19812	2866	281	1085	1122	90	6	2181
0 1 1 1 1. 0 0 0 1. 1 1 0. 1 0. 0	25709	3493	675	1071	1088	88	6	5307
1 1 1 1 1. 0 0 0 1. 1 1 0. 1 0. 0	60645	5927	3760	3419	3443	225	6	5676
0 0 0 1 1. 0 0 1 1. 1 1 0. 1 0. 0	9750	1526	0	68	68	0	5	2192
1 0 0 1 1. 0 0 1 1. 1 1 0. 1 0. 0	19812	2866	281	1085	1122	90	6	2181
0 0 1 1 1. 0 0 1 1. 1 1 0. 1 0. 0	14700	2436	0	18	18	0	6	3536
1 0 1 1 1. 0 0 1 1. 1 1 0. 1 0. 0	16582	2700	152	334	342	42	6	1960
0 1 1 1 1. 0 0 1 1. 1 1 0. 1 0. 0	15834	2540	0	44	44	0	6	3062
1 1 1 1 1. 0 0 1 1. 1 1 0. 1 0. 0	32714	4114	1458	898	914	86	6	3536
0 0 1 1 1. 0 1 1 1. 1 1 0. 1 0. 0	12641	2051	0	24	24	0	5	1825
1 0 1 1 1. 0 1 1 1. 1 1 0. 1 0. 0	30736	4410	164	1038	1066	98	6	1892
0 1 1 1 1. 0 1 1 1. 1 1 0. 1 0. 0	16600	2786	0	16	16	0	6	2002
1 1 1 1 1. 0 1 1 1. 1 1 0. 1 0. 0	42776	5042	1778	973	1010	98	6	2507
0 1 1 1 1. 1 1 1 1. 1 1 0. 1 0. 0	11564	1876	0	16	16	0	5	860
1 1 1 1 1. 1 1 1 1. 1 1 0. 1 0. 0	58532	6426	2349	2296	2332	162	6	3605
0 0 0 1 1. 0 0 1 1. 0 0 1. 1 0. 0	3757	759	0	46	46	0	3	1232
1 0 0 1 1. 0 0 1 1. 0 0 1. 1 0. 0	9537	1610	165	399	409	41	4	1722
0 1 0 1 1. 0 0 1 1. 0 0 1. 1 0. 0	8641	1737	0	52	52	0	6	2734
1 1 0 1 1. 0 0 1 1. 0 0 1. 1 0. 0	19122	2966	283	455	462	52	6	3486
0 1 1 1 1. 0 0 1 1. 0 0 1. 1 0. 0	4892	860	0	18	18	0	4	1207
1 1 1 1 1. 0 0 1 1. 0 0 1. 1 0. 0	16210	1992	669	301	318	42	4	2358
0 1 0 1 1. 1 0 1 1. 0 0 1. 1 0. 0	4822	1024	0	26	26	0	5	1044
1 1 0 1 1. 1 0 1 1. 0 0 1. 1 0. 0	26593	3807	471	983	991	81	6	4196
0 0 1 1 1. 1 0 1 1. 0 0 1. 1 0. 0	2619	583	0	18	18	0	3	726
0 1 1 1 1. 1 0 1 1. 0 0 1. 1 0. 0	8722	1710	0	18	18	0	6	1198
1 1 1 1 1. 1 0 1 1. 0 0 1. 1 0. 0	38299	4485	1844	922	938	86	6	4869
0 1 1 1 1. 1 1 1 1. 0 0 1. 1 0. 0	3001	550	0	8	8	0	3	253
1 1 1 1 1. 1 1 1 1. 0 0 1. 1 0. 0	26632	2871	1348	1128	1152	76	4	3039
0 0 0 1 1. 0 0 1 1. 0 1 1. 1 0. 0	7607	1275	0	18	18	0	4	865
1 0 0 1 1. 0 0 1 1. 0 1 1. 1 0. 0	20209	3501	0	74	74	0	6	2717
1 1 0 1 1. 0 0 1 1. 0 1 1. 1 0. 0	11294	1819	60	60	62	10	5	1443

Table 3 continued

	s	**c**	**m**	**b**	**r**	**i**	**2**	**3**
0 0 1 1 1. 0 0 1 1. 0 1 1. 1 0. 0	8494	1503	0	10	10	0	5	1211
1 0 1 1 1. 0 0 1 1. 0 1 1. 1 0. 0	13897	2419	0	30	30	0	6	2000
1 1 1 1 1. 0 0 1 1. 0 1 1. 1 0. 0	16100	2107	479	61	63	13	5	1635
0 1 0 1 1. 1 0 1 1. 0 1 1. 1 0. 0	16655	2605	140	808	910	88	5	1037
1 1 0 1 1. 1 0 1 1. 0 1 1. 1 0. 0	35121	4955	420	1263	1319	113	6	3575
0 0 1 1 1. 1 0 1 1. 0 1 1. 1 0. 0	8722	1710	0	18	18	0	6	1198
0 1 1 1 1. 1 0 1 1. 0 1 1. 1 0. 0	16698	2910	37	490	529	61	6	960
1 1 1 1 1. 1 0 1 1. 0 1 1. 1 0. 0	36770	4652	1157	923	958	94	6	3555
0 0 1 1 1. 0 1 1 1. 0 1 1. 1 0. 0	6261	1195	0	2	2	0	5	884
1 0 1 1 1. 0 1 1 1. 0 1 1. 1 0. 0	16600	2786	0	16	16	0	6	2002
0 1 1 1 1. 0 1 1 1. 0 1 1. 1 0. 0	8862	1798	0	2	2	0	6	1010
1 1 1 1 1. 0 1 1 1. 0 1 1. 1 0. 0	25651	3445	894	54	55	13	6	2762
0 1 1 1 1. 1 1 1 1. 0 1 1. 1 0. 0	7169	1308	0	22	23	3	5	408
1 1 1 1 1. 1 1 1 1. 0 1 1. 1 0. 0	36709	4721	1392	759	797	79	6	3932
0 0 1 1 1. 0 1 1 1. 1 1 1. 1 0. 0	4061	727	0	4	4	0	4	323
1 0 1 1 1. 0 1 1 1. 1 1 1. 1 0. 0	16614	2872	0	8	8	0	6	942
1 1 1 1 1. 0 1 1 1. 1 1 1. 1 0. 0	17384	2193	741	44	46	14	5	891
0 1 1 1 1. 1 1 1 1. 1 1 1. 1 0. 0	16936	2448	0	884	998	80	5	388
1 1 1 1 1. 1 1 1 1. 1 1 1. 1 0. 0	48428	5548	1784	1687	1721	123	6	2918
0 0 0 1 1. 0 0 1 1. 0 1 1. 1 1. 0	1768	285	0	2	2	0	2	119
1 0 0 1 1. 0 0 1 1. 0 1 1. 1 1. 0	7009	1185	0	4	4	0	4	621
1 1 0 1 1. 0 0 1 1. 0 1 1. 1 1. 0	4692	798	0	2	2	0	4	433
1 1 1 1 1. 0 0 1 1. 0 1 1. 1 1. 0	3406	450	104	10	11	3	3	201
0 1 0 1 1. 1 0 1 1. 0 1 1. 1 1. 0	6139	1092	0	16	18	0	4	450
1 1 0 1 1. 1 0 1 1. 0 1 1. 1 1. 0	11564	1876	0	16	16	0	5	860
0 0 1 1 1. 1 0 1 1. 0 1 1. 1 1. 0	3001	550	0	8	8	0	3	253
0 1 1 1 1. 1 0 1 1. 0 1 1. 1 1. 0	7169	1308	0	22	23	3	5	408
1 1 1 1 1. 1 0 1 1. 0 1 1. 1 1. 0	14516	1899	369	36	40	6	5	897
0 1 1 1 1. 1 1 1 1. 0 1 1. 1 1. 0	1880	357	0	2	2	0	3	89
1 1 1 1 1. 1 1 1 1. 0 1 1. 1 1. 0	8608	1169	283	39	44	7	4	573
0 0 1 1 1. 0 1 1 1. 1 1 1. 1 1. 0	5911	842	0	445	591	51	3	167
1 0 1 1 1. 0 1 1 1. 1 1 1. 1 1. 0	16936	2448	0	884	998	80	5	388
1 1 1 1 1. 0 1 1 1. 1 1 1. 1 1. 0	11448	1387	299	490	583	51	4	234
0 1 1 1 1. 1 1 1 1. 1 1 1. 1 1. 0	9390	1358	0	436	522	42	4	168
1 1 1 1 1. 1 1 1 1. 1 1 1. 1 1. 0	23014	2696	645	916	998	70	5	652
0 0 0 0 1. 0 0 0 1. 0 0 1. 0 1. 1	24823	1815	1863	1506	2017	174	2	4671
1 0 0 0 1. 0 0 0 1. 0 0 1. 0 1. 1	124342	9076	7242	5575	6480	508	4	28910
1 1 0 0 1. 0 0 0 1. 0 0 1. 0 1. 1	61230	5421	3100	2587	3028	283	4	11329
1 1 1 0 1. 0 0 0 1. 0 0 1. 0 1. 1	23983	2671	1081	1432	1669	159	4	2460

Table 3 continued

	s	c	m	b	r	i	2	3
1 1 1 1 1. 0 0 0 1. 0 0 1. 0 1. 1	8875	1076	647	887	1131	103	3	478
0 1 0 0 1. 1 0 0 1. 0 0 1. 0 1. 1	115789	8262	5957	2980	3498	286	4	37887
1 1 0 0 1. 1 0 0 1. 0 0 1. 0 1. 1	104740	8844	5437	4770	5255	377	5	16649
0 0 1 0 1. 1 0 0 1. 0 0 1. 0 1. 1	96935	7785	5354	4330	4547	325	4	21048
0 1 1 0 1. 1 0 0 1. 0 0 1. 0 1. 1	116911	10335	4946	1871	1958	214	6	33272
1 1 1 0 1. 1 0 0 1. 0 0 1. 0 1. 1	79902	8714	3188	3278	3376	290	6	8499
0 0 1 1 1. 1 0 0 1. 0 0 1. 0 1. 1	34661	3613	1849	1820	1941	160	5	4320
0 1 1 1 1. 1 0 0 1. 0 0 1. 0 1. 1	26803	3271	887	824	970	101	5	3456
1 1 1 1 1. 1 0 0 1. 0 0 1. 0 1. 1	39419	4790	1940	2600	2900	226	5	2650
0 1 1 0 1. 1 1 0 1. 0 0 1. 0 1. 1	25039	2296	1123	134	152	20	4	6376
1 1 1 0 1. 1 1 0 1. 0 0 1. 0 1. 1	40559	4404	1679	1676	1810	157	5	2512
0 1 0 1 1. 1 1 0 1. 0 0 1. 0 1. 1	16210	1992	669	301	318	42	4	2358
0 1 1 1 1. 1 1 0 1. 0 0 1. 0 1. 1	16100	2107	479	61	63	13	5	1635
1 1 1 1 1. 1 1 0 1. 0 0 1. 0 1. 1	28499	3616	1216	1324	1445	128	5	1865
0 1 1 1 1. 1 1 1 1. 0 0 1. 0 1. 1	3406	450	104	10	11	3	3	201
1 1 1 1 1. 1 1 1 1. 0 0 1. 0 1. 1	13144	1641	593	804	964	79	4	697
0 0 1 0 1. 0 1 0 1. 1 0 1. 0 1. 1	73901	5490	3856	1686	1976	170	4	25002
1 0 1 0 1. 0 1 0 1. 1 0 1. 0 1. 1	137864	12032	6911	3660	3759	299	6	31500
1 1 1 0 1. 0 1 0 1. 1 0 1. 0 1. 1	38713	4329	1850	1598	1734	146	5	3962
0 0 0 1 1. 0 1 0 1. 1 0 1. 0 1. 1	63361	5315	3613	2117	2282	166	5	17491
0 0 1 1 1. 0 1 0 1. 1 0 1. 0 1. 1	57740	5374	2535	909	1075	111	5	16527
1 0 1 1 1. 0 1 0 1. 1 0 1. 0 1. 1	50377	5963	1707	1218	1243	121	6	6823
1 1 1 1 1. 0 1 0 1. 1 0 1. 0 1. 1	28499	3616	1216	1324	1445	128	5	1865
0 1 1 0 1. 1 1 0 1. 1 0 1. 0 1. 1	58664	5287	2858	1774	1924	150	5	11837
1 1 1 0 1. 1 1 0 1. 1 0 1. 0 1. 1	73248	7610	3447	3603	3645	241	6	4313
0 0 0 1 1. 1 1 0 1. 1 0 1. 0 1. 1	60645	5927	3760	3419	3443	225	6	5676
0 0 1 1 1. 1 1 0 1. 1 0 1. 0 1. 1	38299	4485	1844	922	938	86	6	4869
0 1 1 1 1. 1 1 0 1. 1 0 1. 0 1. 1	36770	4652	1157	923	958	94	6	3555
1 1 1 1 1. 1 1 0 1. 1 0 1. 0 1. 1	47171	5743	2116	2484	2520	180	6	2308
0 0 1 1 1. 0 0 1 1. 1 0 1. 0 1. 1	33954	3116	1569	313	321	41	4	12722
1 0 1 1 1. 0 0 1 1. 1 0 1. 0 1. 1	32714	4114	1458	898	914	86	6	3536
1 1 1 1 1. 0 0 1 1. 1 0 1. 0 1. 1	18188	2346	906	1216	1236	90	4	1002
0 0 1 1 1. 0 1 1 1. 1 0 1. 0 1. 1	38737	3624	1780	86	97	15	5	11980
1 0 1 1 1. 0 1 1 1. 1 0 1. 0 1. 1	42776	5042	1778	973	1010	98	6	2507
0 1 1 1 1. 0 1 1 1. 1 0 1. 0 1. 1	25651	3445	894	54	55	13	6	2762
1 1 1 1 1. 0 1 1 1. 1 0 1. 0 1. 1	33260	4484	1200	890	932	94	6	1406
0 1 1 1 1. 1 1 1 1. 1 0 1. 0 1. 1	14516	1899	369	36	40	6	5	897
1 1 1 1 1. 1 1 1 1. 1 0 1. 0 1. 1	42119	5149	1531	1854	1892	138	6	1660
0 0 1 1 1. 0 1 1 1. 1 1 1. 0 1. 1	8306	810	457	19	23	7	3	1926

Table 3 continued

	s	**c**	**m**	**b**	**r**	**i**	**2**	**3**
1 0 1 1 1. 0 1 1 1. 1 1 1. 0 1. 1	17384	2193	741	44	46	14	5	891
1 1 1 1 1. 0 1 1 1. 1 1 1. 0 1. 1	8658	1228	290	64	68	16	4	234
0 1 1 1 1. 1 1 1 1. 1 1 1. 0 1. 1	11448	1387	299	490	583	51	4	234
1 1 1 1 1. 1 1 1 1. 1 1 1. 0 1. 1	22466	2721	744	1029	1115	84	5	608
0 0 0 1 1. 0 0 1 1. 0 1 1. 1 1. 1	24600	2035	1204	869	1125	97	3	6500
1 0 0 1 1. 0 0 1 1. 0 1 1. 1 1. 1	80060	7344	3567	2530	2840	216	5	16514
1 1 0 1 1. 0 0 1 1. 0 1 1. 1 1. 1	34695	4038	1441	1199	1311	116	5	4043
1 1 1 1 1. 0 0 1 1. 0 1 1. 1 1. 1	13144	1641	593	804	964	79	4	697
0 1 0 1 1. 1 0 1 1. 0 1 1. 1 1. 1	53135	5103	2280	1263	1378	112	5	11973
1 1 0 1 1. 1 0 1 1. 0 1 1. 1 1. 1	58532	6426	2349	2296	2332	162	6	3605
0 0 1 1 1. 1 0 1 1. 0 1 1. 1 1. 1	26632	2871	1348	1128	1152	76	4	3039
0 1 1 1 1. 1 0 1 1. 0 1 1. 1 1. 1	36709	4721	1392	759	797	79	6	3932
1 1 1 1 1. 1 0 1 1. 0 1 1. 1 1. 1	42119	5149	1531	1854	1892	138	6	1660
0 1 1 1 1. 1 1 1 1. 0 1 1. 1 1. 1	8608	1169	283	39	44	7	4	573
1 1 1 1 1. 1 1 1 1. 0 1 1. 1 1. 1	22277	2720	724	1029	1117	84	5	537
0 0 1 1 1. 0 1 1 1. 1 1 1. 1 1. 1	22159	2187	873	775	934	72	4	3800
1 0 1 1 1. 0 1 1 1. 1 1 1. 1 1. 1	48428	5548	1784	1687	1721	123	6	2918
1 1 1 1 1. 0 1 1 1. 1 1 1. 1 1. 1	22466	2721	744	1029	1115	84	5	608
0 1 1 1 1. 1 1 1 1. 1 1 1. 1 1. 1	23014	2696	645	916	998	70	5	652
1 1 1 1 1. 1 1 1 1. 1 1 1. 1 1. 1	42640	4726	1218	2274	2274	132	6	604

i—inverse semigroup (for all $x \in S$ there exists a unique element $y \in S$ such that $xyx = x$ and $yxy = y$)

2—2-nilpotent semigroup ($\mathrm{card}(S^2) = 1$ and $\mathrm{card}(S) > 1$)

3—3-nilpotent semigroup ($\mathrm{card}(S^3) = 1$ and $\mathrm{card}(S^2) > 1$).

We classify ordered 2-nilpotent semigroups because they have been studied in Sect. 3.1. The interest in 3-nilpotent semigroups stems from the observation that almost all finite semigroups are of this type. Distler and Mitchell present in [4], Theorem 2.3, a formula for the number $\alpha(n)$ of all 3-nilpotent semigroups of size n up to isomorphism or anti-isomorphism. The formula provides a lower bound for the number $\beta(n)$ of all semigroups of size n up to isomorphism or anti-isomorphism. Presumably this bound is asymptotic, that is, $\alpha(n)/\beta(n)$ tends to 1 while n tends to infinity. See also Table 7 in [4].

An example follows. Let us consider the table for 5 elements. In the row 1111.000.00.0 and in the column **r** we find the number 313. It means that there are 313 non-isomorphic ordered regular semigroups containing the smallest element and the four remaining elements which are mutually incomparable.

The sums of columns of the tables for 5 and 6 elements (these two tables are parts of Table 3) are presented in Table 4.

We have also constructed and enumerated all non-isomorphic ordered semigroups with 7 elements. The constructed ordered semigroups were not printed or stored in

Table 4 Total numbers of non-isomorphic ordered semigroups

Number of elements	5	6	7
Semigroups	198838	13457454	4207546916
Commutative semigroups	37248	1337698	71748346
Monoids	13371	504634	32113642
Bands	20305	494848	14349957
Regular semigroups	22419	546386	15842224
Inverse semigroups	2886	44275	830584
2-nilpotent semigroups	243	1533	12038
3-nilpotent semigroups	14150	2561653	3215028097

another way (there exist too many of them). We have stored only the enumeration of ordered semigroups of size $n = 7$ in a similar form as it is done for $n \leq 6$ in Table 3. But we do not present here this enumeration (such a table would be a continuation of Table 3) since the corresponding table has $G(7) = 2045$ lines. So, we present only the sums of columns of the table for 7 elements—see Table 4. The table for 7 elements in a similar form as the tables in Table 3 is available on http://physics.ujep.cz/CZ/!usr_files/download-rucne/

We should mention that the numbers in the first row in each table in Table 3 have already been presented in some other works.

1. The number of all non-isomorphic semigroups of size n: [3], Table A.16 (for $n \leq 9$). In addition, see also [17], Sequence A027851(for $n \leq 9$).
2. The number of all non-isomorphic commutative semigroups of size n: [6] (for $n = 9$) and [7] (for $n = 10$). See also [17], Sequence A001426 (for $n \leq 10$), and [3], Table 4.3 (for $n \leq 8$) and Table 5.5 (for $n = 9$).
3. The number of all non-isomorphic monoids of size n: [3], Table A.18 (for $n \leq 10$). See also [17], Sequence A058129 (for $n \leq 8$).
4. The number of all non-isomorphic bands of size n: [3], Table A.16 (for $n \leq 9$). See also [17], Sequence A058112 (for $n \leq 7$).
5. The number of all non-isomorphic inverse semigroups of size n: [3], Table 4.3 (for $n \leq 8$) and Table 5.5 (for $n = 9$). See also [17], Sequence A001428 (for $n \leq 9$).
6. The number of all non-isomorphic 3-nilpotent semigroups of size n: [3], Table A.4 (for $n \leq 17$).

3.4 Linearly ordered semigroups

In this subsection we present the special situation when our algorithm is used for the search of linearly ordered semigroups.

In the special case, when semigroups are ordered linearly (totally), we were able to perform our computation not only for $n \leq 7$ but for $n \leq 10$ (n denotes the number of elements of investigated ordered semigroups).

We have used the following modification of Algorithm 3.5.

Algorithm 3.6 Input: An ordered set $([n], \leq)$ where $1 < 2 < \cdots < n$, for some positive integer n.

Output: $A_1, \dots, A_p \in M_n([n])$ such that $([n], *_{A_1}, \leq), \dots, ([n], *_{A_p}, \leq)$ are all non-isomorphic ordered semigroups of the form $([n], *, \leq)$.

The algorithm works with a matrix $A = (a_{kl}) \in M_n([n] \cup \{0\})$ and with integers $q, i, j, 1 \leq i \leq n, 1 \leq j \leq n$.

1. (Initiation.) $a_{kl} \leftarrow 1$ for all $k, l \in [n]$, $q \leftarrow 1$, $i \leftarrow n$, $j \leftarrow n$, $A_q \leftarrow A$
2. If $a_{ij} < n$ then

$$a_{ij} \leftarrow \begin{cases} a_{ij} + 1 & a_{ij} \neq 0 \\ a_{i-1,j} & a_{ij} = 0,\ i > 1,\ j = 1 \\ a_{i,j-1} & a_{ij} = 0,\ i = 1,\ j > 1 \\ \max\{a_{i-1,j}, a_{i,j-1}\} & a_{ij} = 0,\ i > 1,\ j > 1 \end{cases}$$

   and go to step 4.
3. (We have $a_{ij} = n$.)
   (i) $a_{ij} \leftarrow 0$
   (ii) If $j > 1$ then $j \leftarrow j - 1$ and go to step 2.
   (iii) (We have $j = 1$.) If $i > 1$ then $i \leftarrow i - 1$, $j \leftarrow n$ and go to step 2.
   (iv) (We have $j = i = 1$.) The process is complete and we have all the matrices $A_1, A_2, \dots, A_p$.
4. (Test for associativity.) If $a_{lm} \neq 0$, $a_{k,a_{lm}} \neq 0$, $a_{kl} \neq 0$, $a_{a_{kl},m} \neq 0$ and $a_{k,a_{lm}} \neq a_{a_{kl},m}$ for some $k, l, m \in [n]$ then go to step 2.
5. (i) If $j < n$ then $j \leftarrow j + 1$ and go to step 2.
   (ii) (We have $j = n$.) If $i < n$ then $i \leftarrow i + 1$, $j \leftarrow 1$ and go to step 2.
   (iii) (We have $j = i = n$.) $q \leftarrow q + 1$, $A_q \leftarrow A$ and go to step 2.

In Algorithm 3.6, in comparison with Algorithm 3.5, there is no test for isomorphism, there is no test for compatibility, some changes are made in step 2.

The test for isomorphism becomes unnecessary since the group $\mathrm{Aut}(([n], \leq))$ is trivial.

Let n be a positive integer. The symbol $\leq$ denotes here the usual ordering of natural numbers: $1 \leq 2 \leq 3 \dots$. A matrix $A \in M_n([n])$ is said to be isotone if $j \leq k$ implies $a_{ij} \leq a_{ik}$ and $a_{ji} \leq a_{ki}$ (for any $i, j, k \in [n]$). Then, for any $A \in M_n([n])$, $([n], *_A, \leq)$ is a linearly ordered semigroup if and only if the operation $*_A$ is associative and A is isotone. The modifications made in step 2 guarantee that all computed matrices will be isotone. Thus the test for compatibility is replaced by the new version of step 2.

The numbers of all non-isomorphic linearly ordered semigroups with n elements, $n \leq 10$, are given in Table 5.

Remark 3.7 Let us take a look at Table 5 and compare the numbers of all non-isomorphic linearly ordered bands with the numbers of all non-isomorphic linearly ordered regular semigroups. We see that every linearly ordered regular semigroup with at most 6 elements is a band. There exist (up to isomorphism) exactly two linearly ordered regular semigroups with 7 elements which are not bands. These facts were already proved in 1963 by Saitô in [14], Theorem 3. He also presented the multiplication tables of two linearly ordered regular semigroups which are not bands ([14], Example 2 and Example 4 on page 269). In the context, it is interesting that

Table 5 Numbers of non-isomorphic linearly ordered semigroups

Number of elements	1	2	3	4	5	6	7	8	9	10
Semigroups	1	6	44	386	3852	42640	516791	6817378	98091071	1569786228
Commutative semigroups	1	4	20	114	710	4726	33157	243048	1850817	14590692
Monoids	1	2	8	34	184	1218	9742	92882	1053248	14592054
Bands	1	4	17	82	422	2274	12665	72326	421214	2492112
Regular semigroups	1	4	17	82	422	2274	12667	72348	421414	2493718
Inverse semigroups	1	2	5	14	42	132	429	1430	4862	16796
2-nilpotent semigroups	0	2	3	4	5	6	7	8	9	10
3-nilpotent semigroups	0	0	2	14	83	604	6308	99592	2427926	91291686

Table 6 Numbers of non-isomorphic semigroups, linearly ordered semigroups and ordered semigroups

n	1	2	3	4	5	6	7	8	9	10
$S(n)$	1	5	24	188	1915	28634	1627672	3684030417	105978177936292	?
$LOS(n)$	1	6	44	386	3852	42640	516791	6817378	98091071	1569786228
$OS(n)$	1	11	173	4753	198838	13457454	4207546916	?	?	?

every finite linearly ordered regular semigroup satisfies the identity $x^3 = x^2$ ([14], Theorem 2).

For any positive integer n, let us denote by $S(n)$ the number of all non-isomorphic semigroups with n elements, by *LOS(n)* the number of all non-isomorphic linearly ordered semigroups with n elements and by *OS(n)* the number of all non-isomorphic ordered semigroups with n elements. We compare $S(n)$, *LOS(n)* and *OS(n)* in Table 6. The values $S(n)$ are known for $n \leq 9$ ([3], Table A.16). Up to now, the values *LOS(n)* were known for $n \leq 8$ ([16], 4.1.5, page 61, and [17], Sequence A084965). We have computed the values *LOS(n)* for $n \leq 10$ and *OS(n)* for $n \leq 7$.

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RESEARCH ARTICLE

On varieties of ordered semigroups

Martin Kuřil

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Abstract We use syntactical methods (fully invariant stable quasiorders) to study varieties of ordered semigroups. We prove that the lattice of all varieties of semigroups is embedded into the lattice of all varieties of ordered semigroups. The lattice of all varieties of ordered bands is completely described.

Keywords Variety of ordered semigroups · Fully invariant stable quasiorder · Ordered band

1 Introduction

A structure $(S, \cdot, \leq)$ is called an ordered semigroup if

(i) $(S, \cdot)$ is a semigroup
(ii) $(S, \leq)$ is a partially ordered set
(iii) for any $a, b, c \in S$, $a \leq b$ implies $ca \leq cb$ and $ac \leq bc$.

Let $(S_1, \cdot_1, \leq_1), (S_2, \cdot_2, \leq_2)$ be ordered semigroups. A homomorphism $h : (S_1, \cdot_1, \leq_1) \to (S_2, \cdot_2, \leq_2)$ is any mapping $h : S_1 \to S_2$ which satisfies, for any $a, b \in S_1$,

(i) $h(a \cdot_1 b) = h(a) \cdot_2 h(b)$
(ii) $a \leq_1 b$ implies $h(a) \leq_2 h(b)$.

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M. Kuřil ( )
Department of Mathematics, J.E. Purkyně University, Ceske mladeze 8,
400 96 Usti nad Labem, Czech Republic
e-mail: martin.kuril@ujep.cz

As usual, an ordered semigroup S_2 is said to be a homomorphic image of an ordered semigroup S_1 if there exists a surjective homomorphism of $(S_1, \cdot_1, \leq_1)$ onto $(S_2, \cdot_2, \leq_2)$. Further, an ordered semigroup S_2 is said to be an ordered subsemigroup of an ordered semigroup S_1 if $S_2 \subseteq S_1$ and the following holds for any $a, b \in S_2$:

(i) $a \cdot_2 b = a \cdot_1 b$
(ii) $a \leq_2 b \Longleftrightarrow a \leq_1 b$.

Let $(S_i, \cdot_i, \leq_i), i \in I$, be a set of ordered semigroups. The direct product of ordered semigroups S_i, $i \in I$, is an ordered semigroup $\prod_{i \in I}(S_i, \cdot_i, \leq_i) = (\prod_{i \in I} S_i, \cdot, \leq)$ where, for any $(s_i)_{i \in I}, (t_i)_{i \in I} \in \prod_{i \in I} S_i$,

(i) $(s_i)_{i \in I} \cdot (t_i)_{i \in I} = (s_i \cdot_i t_i)_{i \in I}$
(ii) $(s_i)_{i \in I} \leq (t_i)_{i \in I} \Longleftrightarrow s_i \leq_i t_i$ for all $i \in I$.

Let $\mathcal{C}$ be a class of ordered semigroups. We denote the class of all homomorphic images of ordered semigroups from $\mathcal{C}$ by $\mathrm{H}(\mathcal{C})$, the class of all ordered subsemigroups of ordered semigroups from $\mathcal{C}$ by $\mathrm{S}(\mathcal{C})$ and the class of all direct products of ordered semigroups from $\mathcal{C}$ by $\mathrm{P}(\mathcal{C})$.

A class $\mathcal{V}$ of ordered semigroups is called a variety of ordered semigroups if it is closed with respect to the operators H, S, P, i.e. if $\mathrm{H}(\mathcal{V}) \subseteq \mathcal{V}$, $\mathrm{S}(\mathcal{V}) \subseteq \mathcal{V}$, $\mathrm{P}(\mathcal{V}) \subseteq \mathcal{V}$. The general notion of a variety of ordered algebras has been introduced and studied by Bloom [3].

This paper deals with varieties of ordered semigroups. Let us present an overview of its content.

We denote by **S** (**B**, **NB**, **SL**) the variety of all semigroups (bands, normal bands, semilattices).

If $\mathcal{V}$ is a variety of semigroups then $\mathbf{O}\mathcal{V}$ denotes the ordered semigroup variety consisting of all ordered semigroups $(S, \cdot, \leq)$ such that $(S, \cdot) \in \mathcal{V}$.

If A is a non-empty set then A^+ denotes the free semigroup on A.

As a particular case of a result by Birkhoff [2, VI, Theorem 22], there is a one-to-one correspondence between all varieties of semigroups and all fully invariant congruences on X^+ (where $X = \{x_1, x_2, x_3, \ldots\}$). In Sect. 2, we give an analog to this statement for ordered semigroups. We prove that there is a one-to-one correspondence between all varieties of ordered semigroups and all fully invariant stable quasiorders on X^+. The section has a preparatory character, we present syntactical tools for description of varieties of ordered semigroups.

In Sect. 3 we use syntactical methods (fully invariant stable quasiorders) presented in the previous section to study relations between the lattice $\mathcal{L}(\mathbf{S})$ of all varieties of semigroups and the lattice $\mathcal{L}(\mathbf{OS})$ of all varieties of ordered semigroups. We prove that the lattice $\mathcal{L}(\mathbf{S})$ is in a natural way embedded into the lattice $\mathcal{L}(\mathbf{OS})$. The embedding is not an isomorphism, infinitely many varieties from $\mathcal{L}(\mathbf{OS})$ lie outside of the image of $\mathcal{L}(\mathbf{S})$.

Section 4 forms a central part of the paper. We determine the lattice $\mathcal{L}(\mathbf{OB})$ of all varieties of ordered bands. We use syntactical methods, again, and proceed in the following way: First, we prove that the closed intervals $[\mathbf{SL}, \mathbf{B}]$ and $[\mathbf{OSL}, \mathbf{OB}]$ are isomorphic (see Theorem 4.6). This result provides a description of the interval $[\mathbf{OSL}, \mathbf{OB}]$ since the lattice $\mathcal{L}(\mathbf{B})$ of all varieties of bands has already been determined by Birjukov [1], Fennemore [5] and Gerhard [6]. Further, we prove that $\mathcal{V} \in \mathcal{L}(\mathbf{OB})$, $\neg(\mathcal{V} \in [\mathbf{OSL}, \mathbf{OB}])$ implies $\mathcal{V} \subseteq \mathbf{ONB}$ (see Theorem 4.9). Finally, the lattice $\mathcal{L}(\mathbf{ONB})$ of all varieties of ordered normal bands has already been described by Emery [4].

Our work with fully invariant stable quasiorders on X^+ in Sect. 4 was inspired by Polák's work with fully invariant congruences on U, the free unary semigroup on X, in [9].

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2 Fully invariant stable quasiorders

Let Y be a non-empty set. An inequality is any pair $u \preceq v$ of words $u, v \in Y^+$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. An inequality $u \preceq v$ is satisfied in $(S, \cdot, \leq)$ (or, $(S, \cdot, \leq)$ satisfies the inequality $u \preceq v$) if, for every homomorphism $\varphi : Y^+ \to (S, \cdot)$, $\varphi(u) \leq \varphi(v)$. An inequality is satisfied in a class $\mathcal{C}$ of ordered semigroups if it is satisfied in every ordered semigroup from $\mathcal{C}$.

For a given class $\mathcal{C}$ of ordered semigroups, we put

$$\mathrm{In}_Y(\mathcal{C}) = \{(u, v) \in Y^+ \times Y^+ |\text{ the inequality } u \preceq v \text{ is satisfied in } \mathcal{C}\}.$$

For a given set of inequalities $\Sigma \subseteq Y^+ \times Y^+$, we put

$$\mathrm{Mod}(\Sigma) = \{(S, \cdot, \leq) \in \mathbf{OS}|\ (S, \cdot, \leq) \text{ satisfies all inequalities from } \Sigma\}.$$

Lemma 2.1 *Let $\Sigma \subseteq Y^+ \times Y^+$. Then* $\mathrm{Mod}(\Sigma)$ *is a variety of ordered semigroups.*

Proof It is easy to show the inclusions $\mathrm{H}(\mathrm{Mod}(\Sigma)) \subseteq \mathrm{Mod}(\Sigma)$, $\mathrm{S}(\mathrm{Mod}(\Sigma)) \subseteq \mathrm{Mod}(\Sigma)$, $\mathrm{P}(\mathrm{Mod}(\Sigma)) \subseteq \mathrm{Mod}(\Sigma)$.

A quasiorder (i.e. a reflexive and transitive relation) ρ on a semigroup S is said to be stable if, for every $a, b, c \in S$, $a\rho b$ implies $ca\rho cb$ and $ac\rho bc$. Let ρ be a stable quasiorder on a semigroup $S = (S, \cdot)$. We construct an ordered semigroup $S/\rho = (S/\sim_\rho, \cdot, \leq)$. We define a binary relation $\sim_\rho$ on S in this way: for every $a, b \in S$,

$$a \sim_\rho b \iff a\rho b \text{ and } b\rho a.$$

It can be shown easily that $\sim_\rho$ is a congruence on $(S, \cdot)$. The congruence $\sim_\rho$ determines a semigroup $(S/\sim_\rho, \cdot)$. We define a relation $\leq$ on $S/\sim_\rho$ in this way: for any $a, b \in S$,

$$(a \sim_\rho) \leq (b \sim_\rho) \iff a\rho b.$$

We check easily that the relation $\leq$ on $S/\sim_\rho$ is defined correctly. Further, $(S/\sim_\rho, \cdot, \leq)$ is an ordered semigroup. We will denote it by S/ρ.

Let ρ_i, $i \in I$, be a set of stable quasiorders on a semigroup $S = (S, \cdot)$. Let $\rho = \bigcap_{i \in I} \rho_i$. It is easy to show that ρ is a stable quasiorder on S and S/ρ is isomorphic to an ordered subsemigroup of $\prod_{i\in I} S/\rho_i$. The isomorphism is given by the rule $a \sim_\rho \mapsto (a \sim_{\rho_i})_{i\in I}$, for all $a \in S$.

Let $f : A \to B$ be a mapping, $\tau \subseteq B \times B$. We put

$$f^{-1}(\tau) = \{(u, v) \in A \times A,\ f(u)\tau f(v)\}.$$

Lemma 2.2 *Let $(S, \cdot, \leq)$ be an ordered semigroup, $\varphi : Y^+ \to (S, \cdot)$ be a homomorphism. Then $\varphi^{-1}(\leq)$ is a stable quasiorder on Y^+ and $Y^+/\varphi^{-1}(\leq)$ is isomorphic to an ordered subsemigroup of S.*

Proof We denote $\varphi^{-1}(\leq)$ by ρ. It is easy to prove that ρ is a stable quasiorder on Y^+. Further, let $u \in Y^+$. We put $\psi(u \sim_\rho) = \varphi(u)$. This rule defines correctly a homomorphism $\psi : Y^+/\sim_\rho \to (S, \cdot)$. Now, it is enough to prove that ψ satisfies, for all $u, v \in Y^+$, the condition $(u \sim_\rho) \leq (v \sim_\rho) \Leftrightarrow \psi(u \sim_\rho) \leq \psi(v \sim_\rho)$. But we have $(u \sim_\rho) \leq (v \sim_\rho) \Leftrightarrow u\rho v \Leftrightarrow \varphi(u) \leq \varphi(v) \Leftrightarrow \psi(u \sim_\rho) \leq \psi(v \sim_\rho)$. So, the condition is satisfied.

We present another representation for $\mathrm{In}_Y(\mathcal{C})$. It is easy to see that

$$\mathrm{In}_Y(\mathcal{C}) = \bigcap\{\varphi^{-1}(\leq)|\ (S, \cdot, \leq) \in \mathcal{C},\ \varphi : Y^+ \to (S, \cdot) \text{ is a homomorphism}\}.$$

Denote the class $\{\varphi^{-1}(\leq)|\ (S, \cdot, \leq) \in \mathcal{C},\ \varphi : Y^+ \to (S, \cdot) \text{ is a homomorphism}\}$ by $Q_Y(\mathcal{C})$. So, $\mathrm{In}_Y(\mathcal{C}) = \bigcap Q_Y(\mathcal{C})$. Note that $Q_Y(\mathcal{C})$ is a set, because it is a subclass of the set of all binary relations on Y^+.

A quasiorder ρ on the semigroup Y^+ is said to be fully invariant if, for every $u, v \in Y^+$ and for every endomorphism $\eta : Y^+ \to Y^+$, $u\rho v$ implies $\eta(u)\rho\eta(v)$. The set of all fully invariant stable quasiorders on the semigroup Y^+ will be denoted by $\mathrm{FISQ}(Y^+)$.

Lemma 2.3 *Let $\mathcal{C} \subseteq$ **OS**. Then $\mathrm{In}_Y(\mathcal{C}) \in \mathrm{FISQ}(Y^+)$ and $Y^+/\mathrm{In}_Y(\mathcal{C})$ is isomorphic to an ordered subsemigroup of $\prod_{\rho\in Q_Y(\mathcal{C})} Y^+/\rho$.*

Proof Recall that $\mathrm{In}_Y(\mathcal{C}) = \bigcap Q_Y(\mathcal{C})$. It follows from Lemma 2.2 that $Q_Y(\mathcal{C})$ is a set of stable quasiorders on Y^+. Thus $\bigcap Q_Y(\mathcal{C})$ is a stable quasiorder on Y^+ and $Y^+/\bigcap Q_Y(\mathcal{C})$ is isomorphic to an ordered subsemigroup of $\prod_{\rho\in Q_Y(\mathcal{C})} Y^+/\rho$. It remains to prove that $\bigcap Q_Y(\mathcal{C})$ is fully invariant. Let $(u, v) \in \bigcap Q_Y(\mathcal{C})$. Let $\eta : Y^+ \to Y^+$ be a homomorphism. We have to prove that $(\eta(u), \eta(v)) \in \bigcap Q_Y(\mathcal{C})$. Let $(S, \cdot, \leq) \in \mathcal{C}, \varphi : Y^+ \to (S, \cdot)$ be a homomorphism. We want: $\varphi(\eta(u)) \leq \varphi(\eta(v))$. But it holds since $\varphi \circ \eta : Y^+ \to (S, \cdot)$ is a homomorphism, $(S, \cdot, \leq) \in \mathcal{C}$ and $(u, v) \in \bigcap Q_Y(\mathcal{C})$.

Lemma 2.4 *Let $\rho \in \mathrm{FISQ}(Y^+)$. Then $Y^+/\rho \in \mathrm{Mod}(\rho)$ and $\rho \in Q_Y(\mathrm{Mod}(\rho))$.*

Proof We are going to show that $Y^+/\rho \in \mathrm{Mod}(\rho)$. Let $u, v \in Y^+, u\rho v, \varphi : Y^+ \to Y^+/\sim_\rho$ be a homomorphism. We want to show that $\varphi(u) \leq \varphi(v)$. For any $y \in Y$ let us choose $\vartheta(y) \in Y^+$ in such a way that $\varphi(y) = \vartheta(y) \sim_\rho$. Let $\theta : Y^+ \to Y^+$ be the endomorphism extending the mapping $\vartheta : Y \to Y^+$. Let $y_1, \ldots, y_k \in Y$. Then

$$\begin{aligned}\varphi(y_1 \dots y_k) &= \varphi(y_1)\dots\varphi(y_k)\\ &= (\vartheta(y_1)\sim_\rho)\dots(\vartheta(y_k)\sim_\rho)\\ &= (\vartheta(y_1)\dots\vartheta(y_k))\sim_\rho\\ &= \theta(y_1\dots y_k)\sim_\rho.\end{aligned}$$

We have shown that, for any $w \in Y^+$, $\varphi(w) = \theta(w) \sim_\rho$. So, we want to prove that $(\theta(u) \sim_\rho) \leq (\theta(v) \sim_\rho)$, i.e. $\theta(u)\rho\theta(v)$. But it holds since $\rho \in \mathrm{FISQ}(Y^+)$. It remains to show that $\rho \in Q_Y(\mathrm{Mod}(\rho))$. Let $\psi(u) = u \sim_\rho$ for all $u \in Y^+$. Then $\psi : Y^+ \to Y^+/\sim_\rho$ is a homomorphism. We already know that $Y^+/\rho = (Y^+/\sim_\rho, \cdot, \leq) \in \mathrm{Mod}(\rho)$. Consequently, $\psi^{-1}(\leq) \in Q_Y(\mathrm{Mod}(\rho))$. Let $u, v \in Y^+$. It holds: $(u, v) \in \psi^{-1}(\leq) \Leftrightarrow \psi(u) \leq \psi(v) \Leftrightarrow (u \sim_\rho) \leq (v \sim_\rho) \Leftrightarrow u\rho v$. So, $\psi^{-1}(\leq) = \rho$, $\rho \in Q_Y(\mathrm{Mod}(\rho))$.

Let $\mathcal{C}$ be a class of ordered semigroups. By a free object in $\mathcal{C}$ on a non-empty set Z we mean a pair (S, ι) where $S \in \mathcal{C}$ and $\iota : Z \to S$ is a mapping with the following universal property: for any ordered semigroup $T \in \mathcal{C}$ and any mapping $\vartheta : Z \to T$ there exists a unique homomorphism $\psi : (S, \cdot, \leq) \to (T, \cdot, \leq)$ such that $\psi \circ \iota = \vartheta$. In the cases when the mapping ι is clear we will omit ι and will simply say that S is a free object in $\mathcal{C}$ on Z. Notice that there is, up to isomorphism, at most one free object on a given non-empty set in every class of ordered semigroups.

Theorem 2.5 *Let $\mathcal{V}$ be a variety of ordered semigroups. Then $Y^+/\mathrm{In}_Y(\mathcal{V})$ is a free object in $\mathrm{Mod}(\mathrm{In}_Y(\mathcal{V}))$ on Y and $Y^+/\mathrm{In}_Y(\mathcal{V}) \in \mathcal{V}$. In particular, $Y^+/\mathrm{In}_Y(\mathcal{V})$ is a free object in $\mathcal{V}$ on Y.*

Proof By Lemma 2.3, $Y^+/\mathrm{In}_Y(\mathcal{V})$ is isomorphic to an ordered subsemigroup of $\prod_{\rho\in Q_Y(\mathcal{V})} Y^+/\rho$. According to Lemma 2.2, $Y^+/\rho \in \mathcal{V}$ for each $\rho \in Q_Y(\mathcal{V})$. Since $\mathcal{V}$ is a variety of ordered semigroups, we have $\prod_{\rho\in Q_Y(\mathcal{V})} Y^+/\rho \in \mathcal{V}$, and then $Y^+/\mathrm{In}_Y(\mathcal{V}) \in \mathcal{V}$ as well. It remains to show that $Y^+/\mathrm{In}_Y(\mathcal{V})$ is a free object in $\mathrm{Mod}(\mathrm{In}_Y(\mathcal{V}))$ on Y. Let $(P, \cdot, \leq) \in \mathrm{Mod}(\mathrm{In}_Y(\mathcal{V}))$ and let $\vartheta : Y \to P$. Since Y^+ is the free semigroup over Y, the mapping ϑ has a unique extension to a homomorphism $\widehat{\vartheta} : Y^+ \to P$. But since $(P, \cdot, \leq) \in \mathrm{Mod}(\mathrm{In}_Y(\mathcal{V}))$, given any inequality $(u, v) \in \mathrm{In}_Y(\mathcal{V})$, we have $\widehat{\vartheta}(u) \leq \widehat{\vartheta}(v)$ in P. This implies $\sim_{\mathrm{In}_Y(\mathcal{V})} \subseteq \ker(\widehat{\vartheta})$, so there is a homomorphism $\theta : Y^+/\mathrm{In}_Y(\mathcal{V}) \to P$ satisfying $\theta \circ \pi = \widehat{\vartheta}$, where $\pi : Y^+ \to Y^+/\mathrm{In}_Y(\mathcal{V})$. Note also that θ is order-preserving because $(P, \cdot, \leq) \in \mathrm{Mod}(\mathrm{In}_Y(\mathcal{V}))$. Clearly $\theta \circ \iota = \vartheta$, where $\iota : Y \to Y^+/\mathrm{In}_Y(\mathcal{V})$ is the restriction of π to $Y \subseteq Y^+$, and the uniqueness of $\widehat{\vartheta}$ implies θ is unique as well.

Lemma 2.6 *Let $\rho \in \mathrm{FISQ}(Y^+)$. Then $\rho = \mathrm{In}_Y(\mathrm{Mod}(\rho))$.*

Proof Clearly, $\rho \subseteq \mathrm{In}_Y(\mathrm{Mod}(\rho))$. So, we prove the inclusion $\mathrm{In}_Y(\mathrm{Mod}(\rho)) \subseteq \rho$. By Lemma 2.4, $\rho \in Q_Y(\mathrm{Mod}(\rho))$. Thus $\mathrm{In}_Y(\mathrm{Mod}(\rho)) = \bigcap Q_Y(\mathrm{Mod}(\rho)) \subseteq \rho$.

Lemma 2.7 *Let Y, Z be non-empty sets. Let $s, t \in Y^+$, $u, v \in Z^+$, $s = y_1 \dots y_k$, $t = y_{k+1} \dots y_l$, $u = z_1 \dots z_k$, $v = z_{k+1} \dots z_l$, $y_1, \dots, y_l \in Y$, $z_1, \dots, z_l \in Z$. Let $y_i = y_j$ if and only if $z_i = z_j$, for all $i, j \in \{1, \dots, l\}$. Let $(S, \cdot, \leq)$ be an ordered semigroup. Then $s \preceq t$ is satisfied in $(S, \cdot, \leq)$ if and only if $u \preceq v$ is satisfied in $(S, \cdot, \leq)$.*

Proof Let $s \preceq t$ be satisfied in $(S, \cdot, \leq)$. We will prove that $u \preceq v$ is satisfied in $(S, \cdot, \leq)$. The proof of the opposite implication is similar. Let $\varphi : Z^+ \to (S, \cdot)$ be an arbitrary homomorphism. We want to show that $\varphi(u) \leq \varphi(v)$. There exists a mapping $\vartheta : Y \to S$ such that $\vartheta(y_i) = \varphi(z_i)$ for $i = 1, \ldots, l$. Let $\theta : Y^+ \to (S, \cdot)$ be the homomorphism extending the mapping ϑ. Since $s \preceq t$ is satisfied in $(S, \cdot, \leq)$, we have $\theta(s) \leq \theta(t)$. But

$$\begin{aligned}\theta(s) &= \theta(y_1 \ldots y_k)\\ &= \vartheta(y_1) \ldots \vartheta(y_k)\\ &= \varphi(z_1) \ldots \varphi(z_k)\\ &= \varphi(z_1 \ldots z_k)\\ &= \varphi(u).\end{aligned}$$

Similarly, $\theta(t) = \varphi(v)$. Thus $\varphi(u) \leq \varphi(v)$.

Lemma 2.8 *Let Y, Z be infinite sets, $\mathcal{C}$ be a class of ordered semigroups. Then* $\mathrm{Mod}(\mathrm{In}_Y(\mathcal{C})) = \mathrm{Mod}(\mathrm{In}_Z(\mathcal{C}))$.

Proof We prove that $\mathrm{Mod}(\mathrm{In}_Y(\mathcal{C})) \subseteq \mathrm{Mod}(\mathrm{In}_Z(\mathcal{C}))$. The proof of the inclusion $\mathrm{Mod}(\mathrm{In}_Z(\mathcal{C})) \subseteq \mathrm{Mod}(\mathrm{In}_Y(\mathcal{C}))$ is similar. Let $(S, \cdot, \leq) \in \mathrm{Mod}(\mathrm{In}_Y(\mathcal{C}))$. We will show that $(S, \cdot, \leq) \in \mathrm{Mod}(\mathrm{In}_Z(\mathcal{C}))$. Let $(u, v) \in \mathrm{In}_Z(\mathcal{C})$. We want: $u \preceq v$ is satisfied in $(S, \cdot, \leq)$. Let $u = z_1 \ldots z_k$, $v = z_{k+1} \ldots z_l$, $z_1, \ldots, z_l \in Z$. Since the set Y is infinite, there exist $y_1, \ldots, y_l \in Y$ such that , for all $i, j \in \{1, \ldots, l\}$, $y_i = y_j$ if and only if $z_i = z_j$. Put $s = y_1 \ldots y_k$, $t = y_{k+1} \ldots y_l$. Now, we are going to prove that $(s, t) \in \mathrm{In}_Y(\mathcal{C})$. We have to prove that $s \preceq t$ is satisfied in every ordered semigroup from $\mathcal{C}$. Let $(T, \cdot, \leq) \in \mathcal{C}$. By Lemma 2.7, $s \preceq t$ is satisfied in $(T, \cdot, \leq)$ if and only if $u \preceq v$ is satisfied in $(T, \cdot, \leq)$. But $(u, v) \in \mathrm{In}_Z(\mathcal{C})$, $(T, \cdot, \leq) \in \mathcal{C}$. Thus $u \preceq v$ is satisfied in $(T, \cdot, \leq)$. Consequently, $s \preceq t$ is satisfied in $(T, \cdot, \leq)$. We have proved that $(s, t) \in \mathrm{In}_Y(\mathcal{C})$. Since $(S, \cdot, \leq) \in \mathrm{Mod}(\mathrm{In}_Y(\mathcal{C}))$, $s \preceq t$ is satisfied in $(S, \cdot, \leq)$. Again, using Lemma 2.7, $u \preceq v$ is satisfied in $(S, \cdot, \leq)$.

Lemma 2.9 *Let Y be an infinite set, $\mathcal{V}$ be a variety of ordered semigroups. Then* $\mathcal{V} = \mathrm{Mod}(\mathrm{In}_Y(\mathcal{V}))$.

Proof Clearly, $\mathcal{V} \subseteq \mathrm{Mod}(\mathrm{In}_Y(\mathcal{V}))$. So, we will prove the inclusion $\mathrm{Mod}(\mathrm{In}_Y(\mathcal{V})) \subseteq \mathcal{V}$. Let $(S, \cdot, \leq) \in \mathrm{Mod}(\mathrm{In}_Y(\mathcal{V}))$. Let us consider an infinite set Z with the property $card(Z) \geq card(S)$. According to Theorem 2.5, $Z^+/\mathrm{In}_Z(\mathcal{V})$ is a free object in $\mathrm{Mod}(\mathrm{In}_Z(\mathcal{V}))$ on Z and $Z^+/\mathrm{In}_Z(\mathcal{V}) \in \mathcal{V}$. Let $\vartheta : Z \to S$ be a surjective mapping. Since Y and Z are infinite sets, $\mathrm{Mod}(\mathrm{In}_Y(\mathcal{V})) = \mathrm{Mod}(\mathrm{In}_Z(\mathcal{V}))$ (see Lemma 2.8), thus $(S, \cdot, \leq) \in \mathrm{Mod}(\mathrm{In}_Z(\mathcal{V}))$. Let $\varphi : Z^+/\mathrm{In}_Z(\mathcal{V}) \to (S, \cdot, \leq)$ be a homomorphism satisfying $\varphi \circ \iota = \vartheta$ ($\iota : Z \to Z^+/\sim_{\mathrm{In}_Z(\mathcal{V})}, \iota(z) = z \sim_{\mathrm{In}_Z(\mathcal{V})}$). Necessarily, φ is a surjection. So, $(S, \cdot, \leq) \in \mathcal{V}$.

Theorem 2.10 *Let Y be an infinite set. The rules*

$$\mathcal{V} \mapsto \mathrm{In}_Y(\mathcal{V}),\ \rho \mapsto \mathrm{Mod}(\rho)$$

determine mutually inverse order reversing bijections between all varieties of ordered semigroups and all fully invariant stable quasiorders on Y^+.

Proof The theorem follows immediately from Lemmas 2.1, 2.3, 2.6 and 2.9.

3 Relations between $\mathcal{L}$(S) and $\mathcal{L}$(OS)

Let $\mathcal{C}$ be a class of semigroups. We put

$$\mathbf{O}\mathcal{C} = \{(S, \cdot, \leq) \in \mathbf{OS} \mid (S, \cdot) \in \mathcal{C}\}.$$

For any variety $\mathcal{V}$ of (ordered) semigroups, let $\mathcal{L}(\mathcal{V})$ denote the lattice of all subvarieties in $\mathcal{V}$.

For a given class $\mathcal{C}$ of semigroups, we put

$$\mathrm{Eq}_X(\mathcal{C}) = \{(u, v) \in X^+ \times X^+ \mid \text{the identity } u = v \text{ is satisfied in } \mathcal{C}\}.$$

Theorem 3.1 *Let $\mathcal{V}$ be a semigroup variety. Then*

(i) $\mathbf{O}\mathcal{V}$ *is a variety of ordered semigroups*
(ii) $\mathrm{In}_X(\mathbf{O}\mathcal{V}) = \mathrm{Eq}_X(\mathcal{V})$
(iii) The mapping

$$\mathbf{O} : \mathcal{L}(\mathcal{V}) \to \mathcal{L}(\mathbf{O}\mathcal{V})$$

is an embedding of the lattice of all subvarieties in $\mathcal{V}$ into the lattice of all subvarieties in $\mathbf{O}\mathcal{V}$.

Proof

(i) It is clear.
(ii) $\mathrm{Eq}_X(\mathcal{V}) \subseteq \mathrm{In}_X(\mathbf{O}\mathcal{V})$: It is clear.
$\mathrm{In}_X(\mathbf{O}\mathcal{V}) \subseteq \mathrm{Eq}_X(\mathcal{V})$: Let us denote $\mathrm{Eq}_X(\mathcal{V})$ briefly by $\sim$. We have $(X^+/\sim, \cdot, =) \in \mathbf{O}\mathcal{V}$. Let $(u, v) \in \mathrm{In}_X(\mathbf{O}\mathcal{V})$. Let us consider the homomorphism $\varphi : X^+ \to X^+/\sim$ with $\varphi(w) = w\sim$ $(w \in X^+)$. Since $(X^+/\sim, \cdot, =) \in \mathbf{O}\mathcal{V}$ and $(u, v) \in \mathrm{In}_X(\mathbf{O}\mathcal{V})$, we have $\varphi(u) = \varphi(v)$, $u\sim = v\sim$, $u \sim v$.
(iii) First we prove that the mapping $\mathbf{O}$ is a homomorphism. We use part (ii) and Theorem 2.10. Let $\mathcal{V}_1, \mathcal{V}_2$ be semigroup varieties. Then

$$\begin{aligned}
\mathrm{In}_X(\mathbf{O}(\mathcal{V}_1 \vee \mathcal{V}_2)) &= \mathrm{Eq}_X(\mathcal{V}_1 \vee \mathcal{V}_2) \\
&= \mathrm{Eq}_X(\mathcal{V}_1) \cap \mathrm{Eq}_X(\mathcal{V}_2) \\
&= \mathrm{In}_X(\mathbf{O}\mathcal{V}_1) \cap \mathrm{In}_X(\mathbf{O}\mathcal{V}_2) \\
&= \mathrm{In}_X(\mathbf{O}\mathcal{V}_1 \vee \mathbf{O}\mathcal{V}_2) \\
\mathrm{In}_X(\mathbf{O}(\mathcal{V}_1 \cap \mathcal{V}_2)) &= \mathrm{Eq}_X(\mathcal{V}_1 \cap \mathcal{V}_2) \\
&= \mathrm{Eq}_X(\mathcal{V}_1) \vee \mathrm{Eq}_X(\mathcal{V}_2) \\
&= \mathrm{In}_X(\mathbf{O}\mathcal{V}_1) \vee \mathrm{In}_X(\mathbf{O}\mathcal{V}_2) \\
&= \mathrm{In}_X(\mathbf{O}\mathcal{V}_1 \cap \mathbf{O}\mathcal{V}_2)
\end{aligned}$$

Thus $\mathbf{O}(\mathcal{V}_1 \vee \mathcal{V}_2) = \mathbf{O}\mathcal{V}_1 \vee \mathbf{O}\mathcal{V}_2$, $\mathbf{O}(\mathcal{V}_1 \cap \mathcal{V}_2) = \mathbf{O}\mathcal{V}_1 \cap \mathbf{O}\mathcal{V}_2$.

It remains to show that the mapping $\mathbf{O}$ is injective. Let $\mathcal{V}_1$, $\mathcal{V}_2$ be semigroup varieties, $\mathbf{O}\mathcal{V}_1 = \mathbf{O}\mathcal{V}_2$. Then $\mathrm{In}_X(\mathbf{O}\mathcal{V}_1) = \mathrm{In}_X(\mathbf{O}\mathcal{V}_2)$, $\mathrm{Eq}_X(\mathcal{V}_1) = \mathrm{Eq}_X(\mathcal{V}_2)$ [we have used part (ii)], $\mathcal{V}_1 = \mathcal{V}_2$.

Example 3.2 Let n be an integer, $n > 1$. Let $\mathcal{V}_n = \mathrm{Mod}(x \preceq x^n)$. Suppose that $\mathcal{V}_n = \mathbf{O}\mathcal{U}$ for some semigroup variety $\mathcal{U}$. Consider the structure $(\mathbf{N}, +, \leq)$, where $\mathbf{N} = \{1, 2, 3, \ldots\}$ and $\leq$ is the usual ordering of natural numbers $(1 < 2 < 3 < \cdots)$. It is easy to see that $(\mathbf{N}, +, \leq)$ is an ordered semigroup. Let $a \in \mathbf{N}$. Then $1 \leq n$, $1 \cdot a \leq n \cdot a$, $a \leq n \cdot a$. We see that $(\mathbf{N}, +, \leq)$ satisfies the inequality $x \preceq x^n$. So, $(\mathbf{N}, +, \leq) \in \mathcal{V}_n = \mathbf{O}\mathcal{U}$. Thus $(\mathbf{N}, +) \in \mathcal{U}$. But $\mathrm{In}_X(\mathcal{V}_n) = \mathrm{Eq}_X(\mathcal{U})$ (Theorem 3.1). It gives $(x, x^n) \in \mathrm{Eq}_X(\mathcal{U})$, $(\mathbf{N}, +)$ satisfies the identity $x = x^n$. For $x = 1$ we have $1 = n \cdot 1$, $1 = n$. It is a contradiction. So, for all semigroup varieties $\mathcal{U}$, $\mathbf{O}\mathcal{U} \neq \mathcal{V}_n$. Let r, s be integers, $1 < r < s$. Consider the ordered semigroup $(\mathbf{Z}_{s-1}, +, =)$ where $(\mathbf{Z}_{s-1}, +)$ is the cyclic group of order $s - 1$. Then $(\mathbf{Z}_{s-1}, +, =) \in \mathcal{V}_s$, $\neg((\mathbf{Z}_{s-1}, +, =) \in \mathcal{V}_r)$. Thus $\mathcal{V}_r \neq \mathcal{V}_s$. We have constructed infinitely many ordered semigroup varieties $\mathcal{V}_n$, $n = 2, 3, 4, \ldots$, with the property $\mathcal{V}_n \neq \mathbf{O}\mathcal{U}$ for all semigroup varieties $\mathcal{U}$. Consequently, the embedding $\mathbf{O} : \mathcal{L}(\mathbf{S}) \to \mathcal{L}(\mathbf{OS})$ is no isomorphism.

4 All varieties of ordered bands

In this section we will completely describe the lattice $\mathcal{L}(\mathbf{OB})$ of all varieties of ordered bands.

We adopt the following notation for varieties of semigroups:

T : trivial semigroups (semigroups satisfying the identity $a = x$)
LZ : semigroups of left zeros ($ax = a$)
RZ : semigroups of right zeros ($xa = a$)
SL : samilattices ($x^2 = x, xy = yx$)
LNB : left normal bands ($x^2 = x, axy = ayx$)
RNB : right normal bands ($x^2 = x, xya = yxa$)
ReB : rectangular bands ($x^2 = x, a = axa$)
NB : normal bands ($x^2 = x, axya = ayxa$)
B : bands ($x^2 = x$)
S : semigroups.

We use several operators on words from X^+. Let $u \in X^+$. Then

$\mathsf{c}(u)$ is the set of all variables in u
$\mathsf{h}(u)$ is the first variable of u
$\mathsf{t}(u)$ is the last variable of u
$\mathsf{0}(u)$ is the longest initial segment of u in $card(\mathsf{c}(u)) - 1$ variables
$\mathsf{1}(u)$ is the longest final segment of u in $card(\mathsf{c}(u)) - 1$ variables.

We define some binary relations on X^+. Let $u, v \in X^+$. We put

$(u, v) \in \mathsf{c}$ if and only if $\mathsf{c}(u) = \mathsf{c}(v)$
$(u, v) \in \mathsf{h}$ if and only if $\mathsf{h}(u) = \mathsf{h}(v)$

$(u, v) \in \mathsf{t}$ if and only if $\mathsf{t}(u) = \mathsf{t}(v)$
$(u, v) \in \mathsf{c}'$ if and only if $card(\mathsf{c}(u)) = card(\mathsf{c}(v))$ and $card(\mathsf{c}(u) - \mathsf{c}(v)) \leq 1$.

Let $\rho \subseteq X^+ \times X^+$. We define relations ρ_0, ρ_1 as follows:
$u\rho_0 v$ if and only if there are $p, q \in X^+$ such that $p\rho q$, $\mathsf{0}(p) = u$, $\mathsf{0}(q) = v$
$u\rho_1 v$ if and only if there are $p, q \in X^+$ such that $p\rho q$, $\mathsf{1}(p) = u$, $\mathsf{1}(q) = v$.

Let us denote the fully invariant congruence on X^+ corresponding to the variety of all bands by $\sim_2$ (i.e. $\sim_2 = \mathrm{Eq}_X(\mathbf{B})$).

Recall the basic result concerning the relation $\sim_2$.

Theorem 4.1 *Let $u, v \in X^+$. Then*

$$u \sim_2 v \Longleftrightarrow \mathsf{c}(u) = \mathsf{c}(v) \wedge \underbrace{\mathsf{0}(u) \sim_2 \mathsf{0}(v) \wedge \mathsf{1}(u) \sim_2 \mathsf{1}(v)}_{if\ card(\mathsf{c}(u)) = card(\mathsf{c}(v)) \geq 2}.$$

Proof See e.g. [8, Chapter IV.4].

Theorem 4.2 *Let $\rho \in \mathrm{FISQ}(X^+)$, $\sim_2 \subseteq \rho \subseteq \mathsf{c}$. Then, for any $a, b \in X^+$,*

$$a\rho b \Longleftrightarrow \mathsf{c}(a) = \mathsf{c}(b) \wedge \underbrace{\mathsf{0}(a)\rho_0\mathsf{0}(b) \wedge \mathsf{1}(a)\rho_1\mathsf{1}(b)}_{if\ card(\mathsf{c}(a)) = card(\mathsf{c}(b)) \geq 2}.$$

Proof The implication "$\Rightarrow$" is clear.

Let $a, b \in X^+$, $\mathsf{c}(a) = \mathsf{c}(b)$, and, in the case $card(\mathsf{c}(a)) = card(\mathsf{c}(b)) \geq 2$, $\mathsf{0}(a)\rho_0\mathsf{0}(b)$, $\mathsf{1}(a)\rho_1\mathsf{1}(b)$.

If $card(\mathsf{c}(a)) = card(\mathsf{c}(b)) = 1$, then $a = x^k$, $b = x^l$ for some $x \in X$, k, l positive integers. It is clear that $x^k \sim_2 x$, $x^l \sim_2 x$, thus $a \sim_2 b$, $a\rho b$.

Now, let $card(\mathsf{c}(a)) = card(\mathsf{c}(b)) \geq 2$. There are $c, d \in X^+$, $c\rho d$, $\mathsf{0}(a) = \mathsf{0}(c)$, $\mathsf{0}(b) = \mathsf{0}(d)$. Let $x \in \mathsf{c}(a) - \mathsf{c}(\mathsf{0}(a))$, $y \in \mathsf{c}(b) - \mathsf{c}(\mathsf{0}(b))$, $z \in \mathsf{c}(c) - \mathsf{c}(\mathsf{0}(c))$, $t \in \mathsf{c}(d) - \mathsf{c}(\mathsf{0}(d))$.

Let us consider two cases:

(i) $x = y$
It is $\mathsf{c}(\mathsf{0}(a)) = \mathsf{c}(\mathsf{0}(b))$ and $z = t$, since $\mathsf{c}(c) = \mathsf{c}(d)$. Let us consider the following endomorphism $\eta : X^+ \to X^+$:

$$\eta(z) = x,\ \eta|_{X-\{z\}} = \mathsf{id}_{X-\{z\}}.$$

Put $e = \eta(c)$, $f = \eta(d)$. Then $e\rho f$, $\mathsf{0}(e) = \mathsf{0}(a)$, $\mathsf{0}(f) = \mathsf{0}(b)$, $\mathsf{c}(a) - \mathsf{c}(\mathsf{0}(a)) = \{x\} = \mathsf{c}(e) - \mathsf{c}(\mathsf{0}(e))$, $\mathsf{c}(b) - \mathsf{c}(\mathsf{0}(b)) = \{y\} = \mathsf{c}(f) - \mathsf{c}(\mathsf{0}(f))$.

(ii) $x \neq y$
It follows from $\mathsf{c}(a) = \mathsf{c}(b)$ that $y \in \mathsf{c}(\mathsf{0}(a))$, $x \in \mathsf{c}(\mathsf{0}(b))$. Since $y \notin \mathsf{c}(\mathsf{0}(b))$, $x \notin \mathsf{c}(\mathsf{0}(a))$, $\mathsf{c}(c) = \mathsf{c}(d)$, we have $z = x$ and $t = y$. Put $e = c$, $f = d$. Now, $e\rho f$, $\mathsf{0}(e) = \mathsf{0}(a)$, $\mathsf{0}(f) = \mathsf{0}(b)$, $\mathsf{c}(a) - \mathsf{c}(\mathsf{0}(a)) = \{x\} = \mathsf{c}(e) = \mathsf{c}(\mathsf{0}(e))$, $\mathsf{c}(b) - \mathsf{c}(\mathsf{0}(b)) = \{y\} = \mathsf{c}(f) - \mathsf{c}(\mathsf{0}(f))$.

We have: $a = \mathsf{0}(a)xa'$, $e = \mathsf{0}(a)xe'$, $b = \mathsf{0}(b)yb'$, $f = \mathsf{0}(b)yf'$ $(a', b', e', f' \in X^*)$. Now, $eba\rho fba$. We see that $\mathsf{c}(eba) = \mathsf{c}(a)$, $\mathsf{0}(eba) = \mathsf{0}(a)$, $\mathsf{1}(eba) = \mathsf{1}(a)$,

$\mathsf{c}(fba) = \mathsf{c}(ba)$, $\mathsf{0}(fba) = \mathsf{0}(b) = \mathsf{0}(ba)$, $\mathsf{1}(fba) = \mathsf{1}(a) = \mathsf{1}(ba)$. So, by Theorem 4.1, $a \sim_2 eba$, $fba \sim_2 ba$. Since $\sim_2 \subseteq \rho$, we have $a\rho eba$, $eba\rho fba$, $fba\rho ba$ and, consequently, $a\rho ba$.

Now, we do a dual consideration. There are $c, d \in X^+$, $c\rho d$, $\mathsf{1}(c) = \mathsf{1}(a)$, $\mathsf{1}(d) = \mathsf{1}(b)$. Let $x \in \mathsf{c}(a) - \mathsf{c}(\mathsf{1}(a))$, $y \in \mathsf{c}(b) - \mathsf{c}(\mathsf{1}(b))$, $z \in \mathsf{c}(c) - \mathsf{c}(\mathsf{1}(c))$, $t \in \mathsf{c}(d) - \mathsf{c}(\mathsf{1}(d))$. It can be shown, in a similar way as above, that there are $e, f \in X^+$, $e\rho f$, $\mathsf{1}(e) = \mathsf{1}(a)$, $\mathsf{1}(f) = \mathsf{1}(b)$, $\mathsf{c}(a) - \mathsf{c}(\mathsf{1}(a)) = \{x\} = \mathsf{c}(e) - \mathsf{c}(\mathsf{1}(e))$, $\mathsf{c}(b) - \mathsf{c}(\mathsf{1}(b)) = \{y\} = \mathsf{c}(f) - \mathsf{c}(\mathsf{1}(f))$. We have:

$$\begin{aligned} a &= a'x\mathsf{1}(a), & a' &\in X^* \\ e &= e'x\mathsf{1}(a), & e' &\in X^* \\ b &= b'y\mathsf{1}(b), & b' &\in X^* \\ f &= f'y\mathsf{1}(b), & f' &\in X^*. \end{aligned}$$

Now, $bae\rho baf$. We see that $\mathsf{c}(bae) = \mathsf{c}(ba)$, $\mathsf{0}(bae) = \mathsf{0}(b) = \mathsf{0}(ba)$, $\mathsf{1}(bae) = \mathsf{1}(a) = \mathsf{1}(ba)$, $\mathsf{c}(baf) = \mathsf{c}(b)$, $\mathsf{0}(baf) = \mathsf{0}(b)$, $\mathsf{1}(baf) = \mathsf{1}(b)$. So, by Theorem 4.1, $ba \sim_2 bae$, $baf \sim_2 b$. Since $\sim_2 \subseteq \rho$, we have $ba\rho bae$, $bae\rho baf$, $baf\rho b$ and, consequently, $ba\rho b$. Finally, $a\rho ba$ and $ba\rho b$ gives $a\rho b$.

Theorem 4.3 *Let $\rho \in \mathrm{FISQ}(X^+)$, $\sim_2 \subseteq \rho \subseteq \mathsf{c}$. Then $\rho_0 \cap \mathsf{c} \in \mathrm{FISQ}(X^+)$.*

Proof (i) $\rho_0 \cap \mathsf{c}$ is a quasiorder:
The reflexivity is clear.
The transitivity: Let $a, b, c \in X^+$, $a(\rho_0 \cap \mathsf{c})b$, $b(\rho_0 \cap \mathsf{c})c$. There exist $d, e, f, g \in X^+$, $d\rho e$, $f\rho g$, $\mathsf{0}(d) = a$, $\mathsf{0}(e) = b$, $\mathsf{0}(f) = b$, $\mathsf{0}(g) = c$. Let $x \in \mathsf{c}(d) - \mathsf{c}(\mathsf{0}(d)) = \mathsf{c}(e) - \mathsf{c}(\mathsf{0}(e))$, $y \in \mathsf{c}(f) - \mathsf{c}(\mathsf{0}(f)) = \mathsf{c}(g) - \mathsf{c}(\mathsf{0}(g))$. Let us consider the following endomorphism $\eta : X^+ \to X^+$: $\eta(y) = x$, $\eta|_{X-\{y\}} = \mathsf{id}_{X-\{y\}}$. Put $f' = \eta(f)$, $g' = \eta(g)$. It holds: $f'\rho g'$, $\mathsf{c}(f') = \mathsf{c}(g') = \mathsf{c}(d) = \mathsf{c}(e) = \mathsf{c}(bx)$, $\mathsf{0}(g') = \mathsf{0}(g) = c$, $\mathsf{0}(f') = \mathsf{0}(f) = b$. Now, $dbx\rho ebx$, $f'bx\rho g'bx$. We see that

$$\begin{aligned} \mathsf{c}(ebx) &= \mathsf{c}(e) \cup \mathsf{c}(bx) = \mathsf{c}(bx) \cup \mathsf{c}(bx) = \mathsf{c}(bx), \\ \mathsf{c}(f'bx) &= \mathsf{c}(f') \cup \mathsf{c}(bx) = \mathsf{c}(bx) \cup \mathsf{c}(bx) = \mathsf{c}(bx), \\ \mathsf{0}(ebx) &= \mathsf{0}(e) = b, \quad \mathsf{0}(f'bx) = \mathsf{0}(f') = b, \\ \mathsf{1}(ebx) &= \mathsf{1}(bx), \quad \mathsf{1}(f'bx) = \mathsf{1}(bx). \end{aligned}$$

We have shown that $\mathsf{c}(ebx) = \mathsf{c}(f'bx)$, $\mathsf{0}(ebx) = \mathsf{0}(f'bx)$, $\mathsf{1}(ebx) = \mathsf{1}(f'bx)$. By Theorem 4.1, $ebx \sim_2 f'bx$. Thus $dbx\rho ebx \sim_2 f'bx\rho g'bx$. Since $\sim_2 \subseteq \rho$, we have $dbx\rho g'bx$. Let us notice that $\mathsf{0}(dbx) = \mathsf{0}(d) = a$, $\mathsf{0}(g'bx) = \mathsf{0}(g') = c$. So, $a\rho_0 c$. Clearly, $\mathsf{c}(a) = \mathsf{c}(c)$, which gives $a(\rho_0 \cap \mathsf{c})c$.

(ii) $\rho_0 \cap \mathsf{c}$ is stable: Let $a, b, c \in X^+$, $a(\rho_0 \cap \mathsf{c})b$. There exist $d, e \in X^+$, $d\rho e$, $\mathsf{0}(d) = a$, $\mathsf{0}(e) = b$. Let $x \in \mathsf{c}(d) - \mathsf{c}(\mathsf{0}(d)) = \mathsf{c}(e) = \mathsf{c}(\mathsf{0}(e))$. Let $y \in X$, $\neg(y \in \mathsf{c}(ca))$. Let us consider the following endomorphism $\eta : X^+ \to X^+$: $\eta(x) = y$, $\eta|_{X-\{x\}} = \mathsf{id}_{X-\{x\}}$. Put $d' = \eta(d)$, $e' = \eta(e)$. It is $d'\rho e'$. Further, $cd'\rho ce'$, $\mathsf{0}(cd') = ca$, $\mathsf{0}(ce') = cb$. So, $ca\rho_0 cb$. Now, let us consider the endomorphism

$\eta : X^+ \to X^+$ with $\eta(x) = cy$, $\eta|_{X-\{x\}} = \mathsf{id}_{X-\{x\}}$. Put $d' = \eta(d)$, $e' = \eta(e)$. It is $d'\rho e'$. Further, $\mathsf{0}(d') = ac$, $\mathsf{0}(e') = bc$. Thus $ac\rho_0 bc$. Since $\mathsf{c}(ca) = \mathsf{c}(cb)$, $\mathsf{c}(ac) = \mathsf{c}(bc)$, we have $ca(\rho_0 \cap \mathsf{c})cb$, $ac(\rho_0 \cap \mathsf{c})bc$.

(iii) $\rho_0 \cap \mathsf{c}$ is fully invariant: Let $a, b \in X^+$, $a(\rho_0 \cap \mathsf{c})b$. Let $\eta : X^+ \to X^+$ be an endomorphism. Clearly, $\mathsf{c}(\eta(a)) = \mathsf{c}(\eta(b))$. There exist $c, d \in X^+$, $c\rho d$, $\mathsf{0}(c) = a$, $\mathsf{0}(d) = b$. Let $x \in \mathsf{c}(c) - \mathsf{c}(\mathsf{0}(c)) = \mathsf{c}(d) - \mathsf{c}(\mathsf{0}(d))$. Let $y \in X$, $\neg(y \in \mathsf{c}(\eta(a)))$. Let us consider the following endomorphism $\varphi : X^+ \to X^+$: $\varphi(x) = y$, $\varphi(z) = \eta(z)$ for $z \in X$, $z \neq x$. Then $\varphi(c)\rho\varphi(d)$, $\mathsf{0}(\varphi(c)) = \mathsf{0}(\varphi(ax)) = \mathsf{0}(\varphi(a)\varphi(x)) = \mathsf{0}(\eta(a)y) = \eta(a)$, $\mathsf{0}(\varphi(d)) = \mathsf{0}(\varphi(bx)) = \mathsf{0}(\varphi(b)\varphi(x)) = \mathsf{0}(\eta(b)y) = \eta(b)$. Thus $\eta(a)\rho_0\eta(b)$, $\eta(a)(\rho_0 \cap \mathsf{c})\eta(b)$.

Lemma 4.4 *Let $\rho \in \mathrm{FISQ}(X^+)$, $\sim_2 \subseteq \rho \subseteq \mathsf{c}$. The following conditions are equivalent:*

(i) $\neg(\rho \subseteq \mathsf{h})$
(ii) $xy\rho yxy$
(iii) $\rho_0 = \mathsf{c}'$
(iv) $x\rho_0 y$.

Proof

(i) $\Rightarrow$ (ii): Suppose that $\neg(\rho \subseteq \mathsf{h})$. There are $a, b \in X^+$, $a\rho b$, $\mathsf{h}(a) = x \neq y = \mathsf{h}(b)$. It is $a = xa'$, $b = yb'$ for some $a', b' \in X^*$. We substitute xy for x, yx for y and x for any other variable in $xa'\rho yb'$. We get $xyc\rho yxd$, where $c, d \in X^*$, $\mathsf{c}(c) \subseteq \{x, y\}$, $\mathsf{c}(d) \subseteq \{x, y\}$. Multiplying by xy from the right we get $xycxy\rho yxdxy$. By Theorem 4.1, $xy \sim_2 xycxy$, $yxdxy \sim_2 yxy$. Thus $xy\rho yxy$.

(ii) $\Rightarrow$ (iii): Suppose that $xy\rho yxy$. Clearly, $\rho_0 \subseteq \mathsf{c}'$. Let $a, b \in X^+$, $(a, b) \in \mathsf{c}'$. We will consider two cases.

1. $\mathsf{c}(a) = \mathsf{c}(b)$: Let $z \in X$, $\neg(z \in \mathsf{c}(a))$. Let us substitute az for x, bz for y in $xy\rho yxy$. We get $azbz\rho bzazbz$ and therefore $a\rho_0 b$.
2. $\mathsf{c}(a) \neq \mathsf{c}(b)$: Let $z \in \mathsf{c}(a) - \mathsf{c}(b)$, $z' \in \mathsf{c}(b) - \mathsf{c}(a)$. Let us substitute az' for x, bz for y in $xy\rho yxy$. We get $az'bz\rho bzaz'bz$ and therefore $a\rho_0 b$.

The implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) are clear.

Lemma 4.5 *Let $\rho \in \mathrm{FISQ}(X^+)$, $\sim_2 \subseteq \rho \subseteq \mathsf{c} \cap \mathsf{h}$. The following conditions are equivalent:*

(i) $\neg(\rho_0 \subseteq \mathsf{c})$
(ii) $xyz\rho xzyxyz$
(iii) $\rho_0 = \mathsf{c}' \cap \mathsf{h}$
(iv) $xy\rho_0 xz$.

Proof

(i) $\Rightarrow$ (ii): Suppose that $\neg(\rho_0 \subseteq \mathsf{c})$. There are $a, b \in X^+$, $a\rho b$, $\mathsf{c}(\mathsf{0}(a)) \neq \mathsf{c}(\mathsf{0}(b))$. Let $\mathsf{h}(a) = \mathsf{h}(b) = x$, $y \in \mathsf{c}(\mathsf{0}(a)) - \mathsf{c}(\mathsf{0}(b))$, $z \in \mathsf{c}(\mathsf{0}(b)) - \mathsf{c}(\mathsf{0}(a))$. We have $xa'za''\rho xb'yb''$, where $a', b' \in X^+$, $a'', b'' \in X^*$, $\mathsf{0}(a) = xa'$, $\mathsf{0}(b) = xb'$. We substitute the variable x for any variable different from y and z. We get $czc'\rho dyd'$, where $\mathsf{c}(c) = \{x, y\}$, $\mathsf{c}(d) = \{x, z\}$, $\mathsf{c}(c') \subseteq \{x, y, z\}$, $\mathsf{c}(d') \subseteq \{x, y, z\}$. Further, we substitute xyz for y, xzy for z. We get $xyze\rho xzyf$ for some $e, f \in X^+$, $\mathsf{c}(e) \subseteq \{x, y, z\}$, $\mathsf{c}(f) \subseteq \{x, y, z\}$. We multiply by xyz from the right: $xyzexyz\rho xzyfxyz$. By Theorem 4.1, $xyz \sim_2 xyzexyz$, $xzyfxyz \sim_2 xzyxyz$. Since $\sim_2 \subseteq \rho$, we have $xyz\rho xzyxyz$.

(ii) $\Rightarrow$ (iii): Clearly, $\rho_0 \subseteq \mathsf{c}' \cap \mathsf{h}$. Let $(a, b) \in \mathsf{c}' \cap \mathsf{h}$, $\mathsf{h}(a) = \mathsf{h}(b) = x$. There are $a', b' \in X^\star$, $a = xa'$, $b = xb'$. Let us consider the following two cases:

1. $\mathsf{c}(a) = \mathsf{c}(b)$
   Let $t \in X$, $t \notin \mathsf{c}(a)$. Let us substitute $a't$ for y and $b't$ for z in $xyz\rho xzyxyz$. We get $xa'tb't\rho xb'ta'txa'tb't$. Then $xa'\rho_0 xb'$, $a\rho_0 b$.
2. $\mathsf{c}(a) \neq \mathsf{c}(b)$
   Let $y \in \mathsf{c}(a) - \mathsf{c}(b)$, $z \in \mathsf{c}(b) - \mathsf{c}(a)$. Let us substitute $a'z$ for y and $b'y$ for z in $xyz\rho xzyxyz$. We get $xa'zb'y\rho xb'ya'zxa'zb'y$. Thus $xa'\rho_0 xb'$, i.e. $a\rho_0 b$.

The implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) are clear.

Theorem 4.6 *Let* $\rho \in \mathrm{FISQ}(X^+)$, $\sim_2 \subseteq \rho \subseteq \mathsf{c}$. *Then* $\rho \in \mathrm{FIC}(X^+)$. *Consequently,* $\mathbf{O} : [\mathbf{SL}, \mathbf{B}] \to [\mathbf{OSL}, \mathbf{OB}]$ *is an isomorphism.*

Proof We have to prove that the relation ρ is symmetric. We will prove the following assertion:

$$(\forall a \in X^+)(\forall b \in X^+)(\forall \rho \in \mathrm{FISQ}(X^+))[(\sim_2 \subseteq \rho \subseteq \mathsf{c}) \wedge (a\rho b) \Rightarrow (b\rho a)].$$

We use the induction with respect to $card(\mathsf{c}(a))$.

$$card(\mathsf{c}(a)) = 1$$

Let $b \in X^+$, $\rho \in \mathrm{FISQ}(X^+)$, $\sim_2 \subseteq \rho \subseteq \mathsf{c}$, $a\rho b$. We want to show that $b\rho a$. Let $x \in X$, $a = x$. Then $\mathsf{c}(b) = \{x\}$, $b = x^k$ for some positive integer k. Clearly, $x^k \sim_2 x$, $b \sim_2 a$, $b\rho a$.

$$card(\mathsf{c}(a)) > 1$$

Let $b \in X^+$, $\rho \in \mathrm{FISQ}(X^+)$, $\sim_2 \subseteq \rho \subseteq \mathsf{c}$, $a\rho b$. We want to show that $b\rho a$. By Theorem 4.2,

$$a\rho b \iff \mathsf{c}(a) = \mathsf{c}(b) \wedge \mathsf{0}(a)\rho_0\mathsf{0}(b) \wedge \mathsf{1}(a)\rho_1\mathsf{1}(b)$$
$$b\rho a \iff \mathsf{c}(b) = \mathsf{c}(a) \wedge \mathsf{0}(b)\rho_0\mathsf{0}(a) \wedge \mathsf{1}(b)\rho_1\mathsf{1}(a)$$

We want to show that $\mathsf{0}(b)\rho_0\mathsf{0}(a)$, $\mathsf{1}(b)\rho_1\mathsf{1}(a)$. We will show only that $\mathsf{0}(b)\rho_0\mathsf{0}(a)$ (the rest is a dual consideration). We dinstinguish the following two cases:

1. $\rho \subseteq \mathsf{h}$
   We distinguish the following two cases:

(I) $\rho_0 \subseteq \mathsf{c}$
By Theorem 4.3, $\rho_0 \in \mathrm{FISQ}(X^+)$. Further, $\sim_2 \subseteq \rho_0 \subseteq \mathsf{c}$. We know that $0(a)\rho_0 0(b)$. By the induction hypothesis, $0(b)\rho_0 0(a)$.

(II) $\neg(\rho_0 \subseteq \mathsf{c})$
By Lemma 4.5, $\rho_0 = \mathsf{c}' \cap \mathsf{h}$. Since the relations c' and h are symmetric, we have $0(b)\rho_0 0(a)$.

2. $\neg(\rho \subseteq \mathsf{h})$
By Lemma 4.4, $\rho_0 = \mathsf{c}'$. We know that $0(a)\rho_0 0(b)$. Since the relation c' is symmetric, we have $0(b)\rho_0 0(a)$.

Let $\mathcal{V}$ be a semigroup variety, $\mathbf{SL} \subseteq \mathcal{V} \subseteq \mathbf{B}$. Then $\mathbf{OSL} \subseteq \mathbf{O}\mathcal{V} \subseteq \mathbf{OB}$. Theorem 3.1 asserts that the mapping $\mathbf{O} : \mathcal{L}(\mathbf{B}) \to \mathcal{L}(\mathbf{OB})$ is an embedding. Then $\mathbf{O} : [\mathbf{SL}, \mathbf{B}] \to [\mathbf{OSL}, \mathbf{OB}]$ is an embedding. Let $\mathcal{W}$ be an ordered semigroup variety, $\mathbf{OSL} \subseteq \mathcal{W} \subseteq \mathbf{OB}$. Put $\rho = \mathrm{In}_X(\mathcal{W})$. Then $\rho \in \mathrm{FISQ}(X^+)$, $\mathrm{In}_X(\mathbf{OB}) \subseteq \mathrm{In}_X(\mathcal{W}) \subseteq \mathrm{In}_X(\mathbf{OSL})$, $\mathrm{Eq}_X(\mathbf{B}) \subseteq \rho \subseteq \mathrm{Eq}_X(\mathbf{SL})$, $\sim_2 \subseteq \rho \subseteq \mathsf{c}$. We already know that $\rho \in \mathrm{FIC}(X^+)$. Let $\mathcal{V}$ be a semigroup variety, $\mathrm{Eq}_X(\mathcal{V}) = \rho$. Then $\mathbf{SL} \subseteq \mathcal{V} \subseteq \mathbf{B}$. Since $\mathrm{In}_X(\mathbf{O}\mathcal{V}) = \mathrm{Eq}_X(\mathcal{V}) = \rho = \mathrm{In}_X(\mathcal{W})$, we have $\mathbf{O}\mathcal{V} = \mathcal{W}$. We have just proved that the mapping $\mathbf{O} : [\mathbf{SL}, \mathbf{B}] \to [\mathbf{OSL}, \mathbf{OB}]$ is surjective. Consequently, $\mathbf{O} : [\mathbf{SL}, \mathbf{B}] \to [\mathbf{OSL}, \mathbf{OB}]$ is an isomorphism.

All varieties of ordered normal bands were completely determined by Emery [4]. We are going to present his result.

We will use an abbreviation. For $u, v \in X^+$, we write only $u = v$ instead of $u \preceq v, v \preceq u$.

Theorem 4.7 *The lattice $\mathcal{L}(\mathbf{ONB})$ of varieties of ordered normal bands consists of the following 16 varieties:*

- $\mathcal{V}_1 = \mathrm{Mod}(x^2 = x, axya = ayxa) = \mathbf{ONB}$
- $\mathcal{V}_2 = \mathrm{Mod}(x^2 = x, axa \preceq a)$
- $\mathcal{V}_3 = \mathrm{Mod}(x^2 = x, a \preceq axa)$
- $\mathcal{V}_4 = \mathrm{Mod}(x^2 = x, a = axa) = \mathbf{OReB}$
- $\mathcal{V}_5 = \mathrm{Mod}(x^2 = x, axy = ayx) = \mathbf{OLNB}$
- $\mathcal{V}_6 = \mathrm{Mod}(x^2 = x, ax \preceq a)$
- $\mathcal{V}_7 = \mathrm{Mod}(x^2 = x, a \preceq ax)$
- $\mathcal{V}_8 = \mathrm{Mod}(x^2 = x, ax = a) = \mathbf{OLZ}$
- $\mathcal{V}_9 = \mathrm{Mod}(x^2 = x, xya = yxa) = \mathbf{ORNB}$
- $\mathcal{V}_{10} = \mathrm{Mod}(x^2 = x, xa \preceq a)$
- $\mathcal{V}_{11} = \mathrm{Mod}(x^2 = x, a \preceq xa)$
- $\mathcal{V}_{12} = \mathrm{Mod}(x^2 = x, xa = a) = \mathbf{ORZ}$
- $\mathcal{V}_{13} = \mathrm{Mod}(x^2 = x, xy = yx) = \mathbf{OSL}$
- $\mathcal{V}_{14} = \mathrm{Mod}(x^2 = x, xax \preceq a)$
- $\mathcal{V}_{15} = \mathrm{Mod}(x^2 = x, a \preceq xax)$
- $\mathcal{V}_{16} = \mathrm{Mod}(x^2 = x, a = x) = \mathbf{OT}$.

Proof See [4, Theorem 2.1].

Lemma 4.8 *Let $u, v \in X^+$. It holds:*

1. *If* $\mathsf{h}(u) = \mathsf{h}(v)$, $\mathsf{t}(u) = \mathsf{t}(v)$, $\neg(\mathsf{c}(u) \subseteq \mathsf{c}(v))$, $\neg(\mathsf{c}(v) \subseteq \mathsf{c}(u))$, *then* $\mathrm{Mod}(x^2 = x, u \preceq v) = \mathrm{Mod}(x^2 = x, axa = a)$.
2. *If* $\mathsf{h}(u) = \mathsf{h}(v)$, $\mathsf{t}(u) = \mathsf{t}(v)$, $\neg(\mathsf{c}(u) \subseteq \mathsf{c}(v))$, $\mathsf{c}(v) \subseteq \mathsf{c}(u)$, *then* $\mathrm{Mod}(x^2 = x, u \preceq v) = \mathrm{Mod}(x^2 = x, axa \preceq a)$.
3. *If* $\mathsf{h}(u) = \mathsf{h}(v)$, $\mathsf{t}(u) = \mathsf{t}(v)$, $\mathsf{c}(u) \subseteq \mathsf{c}(v)$, $\neg(\mathsf{c}(v) \subseteq \mathsf{c}(u))$, *then* $\mathrm{Mod}(x^2 = x, u \preceq v) = \mathrm{Mod}(x^2 = x, a \preceq axa)$.
4. *If* $\mathsf{h}(u) = \mathsf{h}(v)$, $\mathsf{t}(u) \neq \mathsf{t}(v)$, $\neg(\mathsf{c}(u) \subseteq \mathsf{c}(v))$, $\neg(\mathsf{c}(v) \subseteq \mathsf{c}(u))$, *then* $\mathrm{Mod}(x^2 = x, u \preceq v) = \mathrm{Mod}(x^2 = x, a = ax)$.
5. *If* $\mathsf{h}(u) = \mathsf{h}(v)$, $\mathsf{t}(u) \neq \mathsf{t}(v)$, $\neg(\mathsf{c}(u) \subseteq \mathsf{c}(v))$, $\mathsf{c}(v) \subseteq \mathsf{c}(u)$, *then* $\mathrm{Mod}(x^2 = x, u \preceq v) = \mathrm{Mod}(x^2 = x, ax \preceq a)$.
6. *If* $\mathsf{h}(u) = \mathsf{h}(v)$, $\mathsf{t}(u) \neq \mathsf{t}(v)$, $\mathsf{c}(u) \subseteq \mathsf{c}(v)$, $\neg(\mathsf{c}(v) \subseteq \mathsf{c}(u))$, *then* $\mathrm{Mod}(x^2 = x, u \preceq v) = \mathrm{Mod}(x^2 = x, a \preceq ax)$.
7. *If* $\mathsf{h}(u) \neq \mathsf{h}(v)$, $\mathsf{t}(u) = \mathsf{t}(v)$, $\neg(\mathsf{c}(u) \subseteq \mathsf{c}(v))$, $\neg(\mathsf{c}(v) \subseteq \mathsf{c}(u))$, *then* $\mathrm{Mod}(x^2 = x, u \preceq v) = \mathrm{Mod}(x^2 = x, a = xa)$.
8. *If* $\mathsf{h}(u) \neq \mathsf{h}(v)$, $\mathsf{t}(u) = \mathsf{t}(v)$, $\neg(\mathsf{c}(u) \subseteq \mathsf{c}(v))$, $\mathsf{c}(v) \subseteq \mathsf{c}(u)$, *then* $\mathrm{Mod}(x^2 = x, u \preceq v) = \mathrm{Mod}(x^2 = x, xa \preceq a)$.
9. *If* $\mathsf{h}(u) \neq \mathsf{h}(v)$, $\mathsf{t}(u) = \mathsf{t}(v)$, $\mathsf{c}(u) \subseteq \mathsf{c}(v)$, $\neg(\mathsf{c}(v) \subseteq \mathsf{c}(u))$, *then* $\mathrm{Mod}(x^2 = x, u \preceq v) = \mathrm{Mod}(x^2 = x, a \preceq xa)$.
10. *If* $\mathsf{h}(u) \neq \mathsf{h}(v)$, $\mathsf{t}(u) \neq \mathsf{t}(v)$, $\neg(\mathsf{c}(u) \subseteq \mathsf{c}(v))$, $\neg(\mathsf{c}(v) \subseteq \mathsf{c}(u))$, *then* $\mathrm{Mod}(x^2 = x, u \preceq v) = \mathrm{Mod}(x^2 = x, a = x)$.
11. *If* $\mathsf{h}(u) \neq \mathsf{h}(v)$, $\mathsf{t}(u) \neq \mathsf{t}(v)$, $\neg(\mathsf{c}(u) \subseteq \mathsf{c}(v))$, $\mathsf{c}(v) \subseteq \mathsf{c}(u)$, *then* $\mathrm{Mod}(x^2 = x, u \preceq v) = \mathrm{Mod}(x^2 = x, xax \preceq a)$.
12. *If* $\mathsf{h}(u) \neq \mathsf{h}(v)$, $\mathsf{t}(u) \neq \mathsf{t}(v)$, $\mathsf{c}(u) \subseteq \mathsf{c}(v)$, $\neg(\mathsf{c}(v) \subseteq \mathsf{c}(u))$, *then* $\mathrm{Mod}(x^2 = x, u \preceq v) = \mathrm{Mod}(x^2 = x, a \preceq xax)$.

Proof See [4, Theorems 3.5., 3.8., 3.9. and 3.12].

Theorem 4.9 *If* $\mathcal{V}$ *is a variety of ordered bands,* $\neg(\mathbf{OSL} \subseteq \mathcal{V})$*, then* $\mathcal{V} \subseteq \mathbf{ONB}$.

Proof We have $\neg(\mathrm{In}_X(\mathcal{V}) \subseteq \mathrm{In}_X(\mathbf{OSL}))$, $\neg(\mathrm{In}_X(\mathcal{V}) \subseteq \mathrm{Eq}_X(\mathbf{SL}))$. There are $u, v \in X^+$, $(u, v) \in \mathrm{In}_X(\mathcal{V})$, $\neg((u, v) \in \mathrm{Eq}_X(\mathbf{SL}))$. Let $S \in \mathcal{V}$. Then $S \in \mathrm{Mod}(x^2 = x, u \preceq v)$. There are four possibilities:

$$\begin{aligned} \mathsf{h}(u) = \mathsf{h}(v), &\quad \mathsf{t}(u) = \mathsf{t}(v) \\ \mathsf{h}(u) = \mathsf{h}(v), &\quad \mathsf{t}(u) \neq \mathsf{t}(v) \\ \mathsf{h}(u) \neq \mathsf{h}(v), &\quad \mathsf{t}(u) = \mathsf{t}(v) \\ \mathsf{h}(u) \neq \mathsf{h}(v), &\quad \mathsf{t}(u) \neq \mathsf{t}(v). \end{aligned}$$

We know that $\mathsf{c}(u) \neq \mathsf{c}(v)$, i.e. $\neg(\mathsf{c}(u) \subseteq \mathsf{c}(v))$, $\neg(\mathsf{c}(v) \subseteq \mathsf{c}(u))$ or $\neg(\mathsf{c}(u) \subseteq \mathsf{c}(v))$, $\mathsf{c}(v) \subseteq \mathsf{c}(u)$ or $\mathsf{c}(u) \subseteq \mathsf{c}(v)$, $\neg(\mathsf{c}(v) \subseteq \mathsf{c}(u))$. Altogether, there are twelve possibilities. By Lemma 4.8 and Theorem 4.7,

$$S \in \mathcal{V}_4 \cup \mathcal{V}_2 \cup \mathcal{V}_3 \cup \mathcal{V}_8 \cup \mathcal{V}_6 \cup \mathcal{V}_7 \cup \mathcal{V}_{12} \cup \mathcal{V}_{10} \cup \mathcal{V}_{11} \cup \mathcal{V}_{16} \cup \mathcal{V}_{14} \cup \mathcal{V}_{15} \subseteq \mathbf{ONB}.$$

(We have used the same notation as in Theorem 4.7)

Remark 4.10 Theorems 4.6 and 4.9 provide a complete description of the lattice $\mathcal{L}(\mathbf{OB})$. Why? Clearly, the lattice $\mathcal{L}(\mathbf{OB})$consists of the closed interval $[\mathbf{OSL}, \mathbf{OB}]$

and of those varieties $\mathcal{V} \in \mathcal{L}(\mathbf{OB})$ for which $\neg(\mathbf{OSL} \subseteq \mathcal{V})$. If $\mathcal{V}$ is a variety of ordered bands, $\neg(\mathbf{OSL} \subseteq \mathcal{V})$, then $\mathcal{V} \in \mathcal{L}(\mathbf{ONB})$ (see Theorem 4.9). The lattice $\mathcal{L}(\mathbf{ONB})$ of all varieties of ordered normal bands was completely described by Emery [4]. We have proved in Theorem 4.6 that the closed intervals $[\mathbf{SL}, \mathbf{B}]$, $[\mathbf{OSL}, \mathbf{OB}]$ are isomorphic (the isomorphism is given by $\mathcal{V} \mapsto \mathbf{O}\mathcal{V}$, for $\mathcal{V} \in [\mathbf{SL}, \mathbf{B}]$). Finally, the lattice of all varieties of bands (consequently, the interval $[\mathbf{SL}, \mathbf{B}]$) was completely described by Birjukov [1], Fennemore [5] and Gerhard [6] (see also [7]).

Consider the embedding $\mathbf{O} : \mathcal{L}(\mathbf{S}) \to \mathcal{L}(\mathbf{OS})$, $\mathcal{V} \mapsto \mathbf{O}\mathcal{V}$ $(\mathcal{V} \in \mathcal{L}(\mathbf{S}))$. Let $M \subseteq \mathcal{L}(\mathbf{S})$. As usual,

$$\mathbf{O}(M) = \{\mathbf{O}\mathcal{V} |\ \mathcal{V} \in M\}.$$

Lemma 4.11

$$\mathcal{L}(\mathbf{OB}) - \mathbf{O}(\mathcal{L}(\mathbf{B})) = \mathcal{L}(\mathbf{ONB}) - \mathbf{O}(\mathcal{L}(\mathbf{NB}))$$

Proof Let $\mathcal{V} \in \mathcal{L}(\mathbf{OB})$, $\mathcal{V} \notin \mathbf{O}(\mathcal{L}(\mathbf{B}))$. Clearly, $\mathcal{V} \notin \mathbf{O}(\mathcal{L}(\mathbf{NB}))$. By Theorem 4.6, $\mathbf{O} : [\mathbf{SL}, \mathbf{B}] \to [\mathbf{OSL}, \mathbf{OB}]$ is an isomorphism. Thus $\mathcal{V} \notin [\mathbf{OSL}, \mathbf{OB}]$. By Theorem 4.9, $\mathcal{V} \subseteq \mathbf{ONB}$, $\mathcal{V} \in \mathcal{L}(\mathbf{ONB})$.

Conversely, let $\mathcal{V} \in \mathcal{L}(\mathbf{ONB})$, $\mathcal{V} \notin \mathbf{O}(\mathcal{L}(\mathbf{NB}))$. Clearly, $\mathcal{V} \in \mathcal{L}(\mathbf{OB})$. Suppose that $\mathcal{V} \in \mathbf{O}(\mathcal{L}(\mathbf{B}))$. So, $\mathcal{V} = \mathbf{O}\mathcal{W}$ for some $\mathcal{W} \in \mathcal{L}(\mathbf{B})$. We know that $\mathcal{V} \subseteq \mathbf{ONB}$. Then $\mathbf{O}\mathcal{W} \subseteq \mathbf{ONB}$. Since $\mathbf{O} : \mathcal{L}(\mathbf{S}) \to \mathcal{L}(\mathbf{OS})$ is an embedding, we have $\mathcal{W} \subseteq \mathbf{NB}$, $\mathcal{W} \in \mathcal{L}(\mathbf{NB})$, $\mathcal{V} \in \mathbf{O}(\mathcal{L}(\mathbf{NB}))$, a contradiction. Thus $\mathcal{V} \notin \mathbf{O}(\mathcal{L}(\mathbf{B}))$.

Now, consider the embedding $\mathbf{O} : \mathcal{L}(\mathbf{B}) \to \mathcal{L}(\mathbf{OB})$. We have the following theorem:

Theorem 4.12 *In the notation of Theorem 4.7,*

$$\mathcal{L}(\mathbf{OB}) - \mathbf{O}(\mathcal{L}(\mathbf{B})) = \{\mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_6, \mathcal{V}_7, \mathcal{V}_{10}, \mathcal{V}_{11}, \mathcal{V}_{14}, \mathcal{V}_{15}\}.$$

Proof Since

$$\mathcal{L}(\mathbf{NB}) = \{\mathbf{T}, \mathbf{LZ}, \mathbf{RZ}, \mathbf{SL}, \mathbf{LNB}, \mathbf{RNB}, \mathbf{ReB}, \mathbf{NB}\},$$

we have

$$\mathbf{O}(\mathcal{L}(\mathbf{NB})) = \{\mathbf{OT}, \mathbf{OLZ}, \mathbf{ORZ}, \mathbf{OSL}, \mathbf{OLNB}, \mathbf{ORNB}, \mathbf{OReB}, \mathbf{ONB}\}.$$

Then, by Theorem 4.7,

$$\mathcal{L}(\mathbf{ONB}) - \mathbf{O}(\mathcal{L}(\mathbf{NB})) = \{\mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_6, \mathcal{V}_7, \mathcal{V}_{10}, \mathcal{V}_{11}, \mathcal{V}_{14}, \mathcal{V}_{15}\}.$$

We use Lemma 4.11 and the proof is complete.

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RESEARCH ARTICLE

On Varieties of Semilattice-Ordered Semigroups

Martin Kuřil and Libor Polák*

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Abstract

We initiate here a systematic equational classification of semilattice-ordered semigroups. We start with various examples. Then we consider semilattice-ordered semigroups satisfying the identity $x^r = x$. Finally, we reduce the classification to finding certain operators on relatively free semigroups.

Keywords: Variety, semilattice-ordered semigroup, admissible closure operator.

MSC 2000 Classification: 20M07 Varieties of semigroups, 06F05 Ordered semigroups and monoids.

1. Introduction

The investigation of semilattice-ordered semigroups is justified by the examples and theorems in Section 2; they concern endomorphisms of semilattices, power structures of semigroups and sets of binary relations. Recently the second author [8] introduced the so-called syntactic semiring of a given formal language. He proved a modification of the Eilenberg theorem relating certain classes of languages and pseudovarieties of idempotent semirings (= semilattice-ordered monoids). It becomes the next reason for studying the varieties of semilattice-ordered semigroups and monoids.

Section 3 is devoted to the study of free semilattice-ordered semigroups satisfying $x^r = x$. A free object on a non-empty set X in the variety of all semilattice-ordered semigroups satisfying $x^r = x$ is described as a quotient semilattice-ordered semigroup $\mathsf{F}(S_r)/\varrho$ where S_r is a free object on X in the variety of all semigroups satisfying $x^r = x$ and $\mathsf{F}(S)$ is the semilattice-ordered semigroup of all non-empty finite subsets of S with the natural multiplication and join (see also Example 2.2). An important role is played by a special closure operator $[\,]$ on S_r (namely, $A \mathrel{\varrho} B$ if and only if $[A] = [B]$ for any non-empty finite subsets A, B of S_r). This motivates the investigation of certain closure operators on relatively free semigroups in the next section. An other relationship to languages is also mentioned there.

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In Section 4 we define an admissible closure operator which turn out to be the central notion of the paper. Let X^+ be a free semigroup on the set $X = \{x_1, x_2, \ldots\}$, let ρ be a fully invariant congruence on X^+. A ρ-admissible closure operator is a closure operator on X^+/ρ satisfying some additional conditions. We know that that $X^\square = \mathsf{F}(X^+)$ is a free semilattice-ordered semigroup on the set X (see Theorem 2.5). We will prove that there exists a one-to-one correspondence between all fully invariant congruences on $X^\square$ and all ordered pairs $(\rho, [\,])$ where ρ is a fully invariant congruence on X^+ and $[\,]$ is a ρ-admissible closure operator. As is well-known, the lattice of all varieties of semilattice-ordered semigroups is dually isomorphic to the lattice of all fully invariant congruences on $X^\square$. Therefore the study of varieties of semilattice-ordered semigroups is reduced to the description of all admissible closure operators on relatively free semigroups.

Further, in Section 5, we introduce an auxiliary notion of an admissible partial order which is helpful for finding admissible closure operators. In more detail, let ρ be a fully invariant congruence on X^+ and let $[\,]$ be a ρ-admissible closure operator. If we put, for $u, v \in X^+/\rho$, $u \leq v$ if and only if $u \in [\{v\}]$, then the relation $\leq$ is a ρ-admissible partial order.

Emery in [3] has described the lattice of varieties of ordered normal bands. The authors in [6] have used their results from the present paper for finding all varieties of semilattice-ordered normal bands and for solving the word problems in each of them. These results are partially announced in the last section. The lattice of all varieties mentioned above is also described by Ghosh, Pastijn and Zhao in [12].

2. Examples and representations

A structure $(A, \cdot, \vee)$ is called a *semilattice-ordered semigroup* if

(i) $(A, \cdot)$ is a semigroup,

(ii) $(A, \vee)$ is a semilattice,

(iii) for any $a, b, c \in A$, $a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$.

A structure $(A, \cdot, \leq)$ is called an *ordered semigroup* if

(i) $(A, \cdot)$ is a semigroup,

(ii) $(A, \leq)$ is a (partially) ordered set,

(iii) for any $a, b, c \in A$, $a \leq b$ implies $ca \leq cb$ and $ac \leq bc$.

If $(A, \cdot, \vee)$ is a semilattice-ordered semigroup and we define, for any $a, b \in A$, $a \leq b$ if and only if $a \vee b = b$, then the structure $(A, \cdot, \leq)$ is an ordered semigroup.

At first we present certain basic and natural examples of semilattice-ordered semigroups.

Example 2.1. [[2], XIV.4., Example 4] Let $(A,\vee)$ be a semilattice. Then the set $\mathsf{End}(A,\vee)$ of all endomorphisms of $(A,\vee)$ with the operations of composition and join ($(\alpha\vee\beta)(a)=\alpha(a)\vee\beta(a)$ for $\alpha,\beta\in\mathsf{End}(A,\vee)$, $a\in A$) is a semilattice-ordered semigroup.

The set of all subsets of a set A will be denoted by $\mathsf{P}(A)$.

Example 2.2. [[2], XIV.1., Example 2] Let S be a semigroup. We put, for any $Q,R\in\mathsf{P}(S)$,

$$Q\cdot R=\{qr \mid q\in Q, r\in R\}.$$

Then $(\mathsf{P}(S),\cdot,\cup)$ and $(\mathsf{F}(S),\cdot,\cup)$ are semilattice-ordered semigroups. Here, as usual, $\cup$ denotes the set-theoretical union.

Example 2.3. Let A be a set. Then the set $\mathsf{Rel}(A)$ of all binary relations on A with the operations of composition and union is a semilattice-ordered semigroup.

The importance of the given examples follows from the next three theorems.

Theorem 2.4. *Every semilattice-ordered semigroup is isomorphic to a substructure of a semilattice-ordered semigroup constructed in Example 2.1.*

Proof. Let $(A,\cdot,\vee)$ be a semilattice-ordered semigroup. We add to A a new element ε and we denote $A^\varepsilon=A\cup\{\varepsilon\}$. we also xtend the operations $\cdot$ and $\vee$ to A^ε as follows: for any $a\in A$, $\varepsilon\cdot\varepsilon=\varepsilon\cdot a=a\cdot\varepsilon=\varepsilon$, $\varepsilon\vee\varepsilon=\varepsilon$, $\varepsilon\vee a=a\vee\varepsilon=a$. As well-known $(A^\varepsilon,\cdot,\vee)$ is a semilattice-ordered semigroup.

Define a binary operation $\vee$ on the set $\{0,1\}$ in the following way: $0\vee 0=0$, $1\vee 1=1$, $0\vee 1=1\vee 0=1$. Clearly, $(\{0,1\},\vee)$ is a semilattice. Put

$$(B,\vee)=(A^\varepsilon,\vee)\times(\{0,1\},\vee).$$

For any $a\in A$, define a mapping $F(a)\colon B\to B$ as follows: Let $x\in A^\varepsilon$, $i\in\{0,1\}$. We put

$$F(a)((x,i))=\begin{cases}(ax,0) & \text{for } i=0\\ (ax\vee a,0) & \text{for } i=1.\end{cases}$$

We will show: $F(a)\in\mathsf{End}(B,\vee)$ for any $a\in A$. Let $x,y\in A^\varepsilon$, $i,j\in\{0,1\}$. We distinguish four cases:

(a) If $i=j=0$ then $F(a)((x,0)\vee(y,0))=F(a)((x\vee y,0))=(ax\vee ay,0)$ and $F(a)((x,0))\vee F(a)((y,0))=(ax,0)\vee(ay,0)=(ax\vee ay,0)$.

(b) If $i=0,j=1$ then $F(a)((x,0)\vee(y,1))=F(a)((x\vee y,1))=(ax\vee ay\vee a,0)$ and $F(a)((x,0))\vee F(a)((y,1))=(ax,0)\vee(ay\vee a,0)=(ax\vee ay\vee a,0)$.

(c) $i=1,j=0$ is similar to (b).

(d) If $i=j=1$ then $F(a)((x,1)\vee(y,1))=F(a)((x\vee y,1))=(ax\vee ay\vee a,0)$ and $F(a)((x,1))\vee F(a)((y,1))=(ax\vee a,0)\vee(ay\vee a,0)=(ax\vee ay\vee a,0)$.

Now, we know that F maps A to $\mathsf{End}(B,\vee)$. To finish the proof of the theorem it remains to show that F is an injective homomorphism from $(A,\cdot,\vee)$ to $(\mathsf{End}(B,\vee),\circ,\vee)$.

(a) F is a homomorphism: Let $a,b \in A$. We want to show that $F(ab) = F(a)\circ F(b)$. Let $x \in A^\varepsilon$, $i \in \{0,1\}$.

For $i = 0$ we have $F(ab)((x,0)) = (abx,0)$, $(F(a)\circ F(b))((x,0)) = F(a)(F(b)((x,0))) = F(a)((bx,0)) = (abx,0)$, and for $i = 1$ we have $F(ab)((x,1)) = (abx \vee ab,0)$, $(F(a)\circ F(b))((x,1)) = F(a)(F(b)((x,1))) = F(a)((bx\vee b,0)) = (abx \vee ab,0)$. Further we want to show that $F(a\vee b) = F(a)\vee F(b)$. Let $x \in A^\varepsilon$, $i \in \{0,1\}$.

For $i=0$ we have $F(a\vee b)((x,0)) = ((a\vee b)x,0)$, $(F(a)\vee F(b))((x,0)) = F(a)((x,0))\vee F(b)((x,0)) = (ax,0)\vee(bx,0) = ((a\vee b)x,0)$, and for $i=1$ we have $F(a\vee b)((x,1)) = ((a\vee b)x\vee a\vee b,0)$, $(F(a)\vee F(b))((x,1)) = F(a)((x,1))\vee F(b)((x,1)) = (ax\vee a,0)\vee(bx\vee b,0) = ((a\vee b)x\vee a\vee b,0)$.

(b) F is an injective mapping: Let $a,b \in A$, $a \neq b$. We want to show that $F(a)\neq F(b)$. It holds: $F(a)((\varepsilon,1)) = (a\varepsilon\vee a,0) = (a,0)$ and $F(b)((\varepsilon,1)) = (b\varepsilon\vee b,0) = (b,0)\neq(a,0)$. ∎

Theorem 2.5. *The structure $(X^\square,\cdot,\cup)$ together with the embedding κ: $x \to \{x\}$, $x\in X$, is a free object on the set X in the variety of all semilattice-ordered semigroups.*

Proof. Let $(A,\cdot,\vee)$ be a semilattice-ordered semigroup, λ: $X\to A$. Define a mapping ψ: $X^\square \to A$ in the following way: for any $Q\in X^\square$, $\psi(Q) = \bigvee_{q\in Q}\lambda(q)$. It is easy to check that ψ is the unique homomorphism from $(X^\square,\cdot,\cup)$ to $(A,\cdot,\vee)$ such that $\psi\circ\kappa = \lambda$. ∎

A semilattice-ordered semigroup $(A,\cdot,\vee)$ is said to be *representable with binary relations* if there is a set B and an injective homomorphism f: $(A,\cdot,\vee)\to(\mathsf{Rel}(B),\circ,\cup)$.

Recall here that a semilattice $(A,\vee)$ is called *distributive* if for any $x,y,z\in A$, $x\le y\vee z \Longrightarrow x = y'\vee z'$ for some $y'\le y, z'\le z$.

Theorem 2.6 (H. Andréka [1], Theorem 1). *Let $(A,\cdot,\vee)$ be a semilattice-ordered se-migroup such that $(A,\vee)$ is distributive. Then $(A,\cdot,\vee)$ is representable with binary relations.*

The following result is a special case; we provide here a direct and elementary proof.

Theorem 2.7. *Let S be a semigroup. Then the semilattice-ordered semigroup $(\mathsf{P}(S),\cdot,\cup)$ is representable with binary relations.*

Proof. Let $A\in\mathsf{P}(S)$. Define a relation $f(A)$ on S^1 in the following way: for any $x,y\in S^1$,

$$x\ f(A)\ y \iff x = ay \text{ for some } a\in A.$$

We will show that the mapping $f\colon \mathsf{P}(S) \to \mathsf{Rel}(S^1)$ is an injective homomorphism from $(\mathsf{P}(S), \cdot, \cup)$ to $(\mathsf{Rel}(S^1), \circ, \cup)$. Let $A, B \in \mathsf{P}(S)$.

(a) $f(AB) = f(A) \circ f(B)$: Let $x, y \in S^1$, $xf(AB)y$. There is $c \in AB$ such that $x = cy$. But $c = ab$ for some $a \in A$, $b \in B$. Then $x = aby$. Put $z = by$. It holds: $xf(A)z$, $zf(B)y$. So, $x(f(A) \circ f(B))y$. Let $x, y \in S^1$, $x(f(A) \circ f(B))y$. There is $z \in S^1$ such that $xf(A)z$, $zf(B)y$. Then $x = az$, $z = by$ for some $a \in A$, $b \in B$. Now, $x = az = aby$ and $xf(AB)y$ since $ab \in AB$.

(b) $f(A \cup B) = f(A) \cup f(B)$: Let $x, y \in S^1$, $xf(A \cup B)y$. There is $c \in A \cup B$ such that $x = cy$. If $c \in A$ then $xf(A)y$. If $c \in B$ then $xf(B)y$. In any case, $x(f(A) \cup f(B))y$. Let $x, y \in S^1$, $x(f(A) \cup f(B))y$. If $xf(A)y$ then $x = ay$ for some $a \in A$. If $xf(B)y$ then $x = by$ for some $b \in B$. In any case $xf(A \cup B)y$ since $a, b \in A \cup B$.

(c) $A \neq B \Longrightarrow f(A) \neq f(B)$: Let $A \neq B$. Then $A \not\subseteq B$ or $B \not\subseteq A$. Let, for instance, $A \not\subseteq B$. There is $a \in A$ such that $a \notin B$. Clearly, $af(A)1$. We will show that $\neg(af(B)1)$. Suppose that $af(B)1$. Then there exists $b \in B$ with $a = b \cdot 1 = b \in B$ which is a contradiction. ∎

Corollary 2.8. *The free semilattice-ordered semigroup over X is representable by binary relations.*

Evidently, each semigroup can be ordered. If $(S, \cdot)$ is a semigroup and $=$ is the equality relation on S, then $(S, \cdot, =)$ is an ordered semigroup. On the other hand, there exist semigroups which cannot be semilattice-ordered. Recall that a semigroup is said to be a *regular band* if it is a band (i.e. a semigroup of idempotents) satisfying the identity $xyxzx = xyzx$.

The following theorem appears also in Pastijn and Zhao [7], Theorem 2.3.

Theorem 2.9. *If $(S, \cdot, \vee)$ is a semilattice-ordered band then $(S, \cdot)$ is a regular band.*

Proof. Let $a, b, c \in S$. Then

$$\begin{aligned} abc &= ababc \vee abcabc = ab(a \vee ca)bc = ab(a \vee ca)^2bc = ab(a \vee aca \vee ca)bc \\ &= \underline{abab}c \vee abacabc \vee \underline{abcabc} = abc \vee abacabc, \end{aligned}$$

$$\begin{aligned} abacabc &= ababacabc \vee abacacabc = aba(b \vee c)acabc = aba(b \vee c)^2acabc \\ &= aba(b \vee bc \vee cb \vee c)acabc \\ &= \underline{abab}acabc \vee \underline{abab}\,\underline{caca}bc \vee a\underline{bacbac}abc \vee aba\underline{caca}bc \\ &= abacabc \vee \underline{abcabc} \vee abacabc \vee abacabc = abc \vee abacabc \end{aligned}$$

We have shown that $abc = abacabc$. Similarly, $abc = abcacbc$, and so $abacabc = abcacbc$. Finally, notice that a band S is regular if and only if it satisfies the identity $xyxzxyz = xyzxzyz$. ∎

3. Free semilattice-ordered semigroups satisfying $x^r = x$

Let us fix an integer $r \geq 2$. We denote by $\mathcal{S}(x^r = x)$ the class of all semigroups satisfying the identity $x^r = x$ and by $\mathcal{SLOS}(x^r = x)$ the class of all semilattice-ordered semigroups satisfying this identity.

Let S be a semigroup. A subset $M \subseteq S$ is said to be r-*closed* if it satisfies:

$$(\forall\ p, q \in S^1, u_1, u_2, \ldots, u_r \in S)(pu_1q, pu_2q, \ldots, pu_rq \in M$$
$$\text{implies } pu_1u_2 \ldots u_rq \in M).$$

Note that the idea of the 2-closed subset appears also in Zhao [10]. The following relationship to regular languages should be further explored (see [8]):

$$L \subseteq A^* \text{ is } r\text{-closed} \iff \text{the syntactic semiring of } L \text{ satisfies } x_1 \ldots x_r \leq x_1 \vee \ldots \vee x_r.$$

Since S is clearly r-closed and the intersection of any non-empty collection of r-closed subsets is also r-closed, we get a closure operator on S:

$$[\ ]\colon\ \mathsf{P}(S) \to \mathsf{P}(S),\ A \mapsto [A]$$

where $[A]$ is the smallest r-closed subset in S containing A.

Now, we are going to define a binary relation ϱ on $\mathsf{F}(S)$ by

$$A\ \varrho\ B \iff [A] = [B].$$

Clearly, ϱ is an equivalence relation. Moreover, the following theorem holds.

Theorem 3.1. *If $S \in \mathcal{S}(x^r = x)$ then ϱ is a congruence on $(\mathsf{F}(S), \cdot, \cup)$ and $\mathsf{F}(S)/\varrho \in \mathcal{SLOS}(x^r = x)$.*

The theorem for the case $n = 2$ appears already in [10]. Before proving the theorem we formulate two lemmas.

Lemma 3.2. *Let $S \in \mathcal{S}(x^r = x)$. We define, for arbitrary $A \in \mathsf{P}(S)$, the sets $A^{(0)}, A^{(1)}, \ldots$ recursively in the following way: $A^{(0)} = A$,*

$$A^{(n)} = \{pu_1 \ldots u_rq \mid p, q \in S^1, u_1, \ldots, u_r \in S, pu_1q, \ldots, pu_rq \in A^{(n-1)}\},$$
$$n = 1, 2, \ldots.$$

Then, for any $A, B \in \mathsf{P}(S)$,

(i) $A^{(0)} \subseteq A^{(1)} \subseteq A^{(2)} \subseteq \cdots$,

(ii) $A \subseteq B$ *implies* $A^{(n)} \subseteq B^{(n)}$ *for* $n = 0, 1, 2, \ldots$,

(iii) $[A] = \bigcup_{n=0}^{\infty} A^{(n)}$.

Proof. (i): Let $n \geq 1$. We will show that $A^{(n-1)} \subseteq A^{(n)}$. Let $a \in A^{(n-1)}$. Put $p = q = 1$, $u_1 = \cdots = u_r = a$. Clearly, $pu_1q, \ldots, pu_rq \in A^{(n-1)}$ and $a = a^r = 1\underbrace{a \ldots a}_{r}1 = pu_1 \ldots u_rq \in A^{(n)}$.

(ii): Let $A \subseteq B$. We will prove inductively that $A^{(n)} \subseteq B^{(n)}$.

$n = 0$: $A^{(0)} \subseteq B^{(0)}$ since $A^{(0)} = A$, $B^{(0)} = B$.

$n \geq 1$: Suppose that $A^{(n-1)} \subseteq B^{(n-1)}$. Let $x \in A^{(n)}$. Then $x = pu_1 \ldots u_rq$ for some $p, q \in S^1, u_1, \ldots, u_r \in S$, $pu_1q, \ldots, pu_rq \in A^{(n-1)} \subseteq B^{(n-1)}$. It follows from the definition of $B^{(n)}$ that $x \in B^{(n)}$.

(iii): (a): $\bigcup_{n=0}^{\infty} A^{(n)}$ is r-closed:

Let $p, q \in S^1, u_1, \ldots, u_r \in S$, $pu_1q, \ldots, pu_rq \in \bigcup_{n=0}^{\infty} A^{(n)}$. There exists n_0 such that $pu_1q, \ldots, pu_rq \in A^{(n_0)}$ (see the part (i) of this lemma). Then $pu_1 \ldots u_rq \in A^{(n_0+1)} \subseteq \bigcup_{n=0}^{\infty} A^{(n)}$.

(b): $A \subseteq \bigcup_{n=0}^{\infty} A^{(n)}$ since $A = A^{(0)}$.

(c): Let $M \subseteq S$ is r-closed, $A \subseteq M$. We will show that $\bigcup_{n=0}^{\infty} A^{(n)} \subseteq M$. We will prove inductively that $A^{(n)} \subseteq M$ for $n \geq 0$: The case $n = 0$ follows from $A^{(0)} = A \subseteq M$. So let $n \geq 1$ and suppose that $A^{(n-1)} \subseteq M$. Let $x \in A^{(n)}$. Then $x = pu_1 \ldots u_rq$ for some $p, q \in S^1, u_1, \ldots, u_r \in S$, $pu_1q, \ldots, pu_rq \in A^{(n-1)} \subseteq M$. Since M is r-closed, $x \in M$. ∎

Lemma 3.3. *Let $S \in \mathcal{S}(x^r = x)$. For any natural number n and any $A, B, C \in \mathsf{P}(S)$, if $A \subseteq B^{(n)}$ then $AC \subseteq (BC)^{(n)}$.*

Proof. By induction with respect to n.

$n = 0$: Clearly, if $A \subseteq B$ then $AC \subseteq BC$.

$n \geq 1$: Let $A \subseteq B^{(n)}$. Choose arbitrary elements $a \in A$, $c \in C$. We want to show that $ac \in (BC)^{(n)}$. Since $a \in B^{(n)}$, we have $a = pu_1 \ldots u_rq$ for some $p, q \in S^1, u_1, \ldots, u_r \in S$, with $pu_1q, \ldots, pu_rq \in B^{(n-1)}$. Clearly, $\{pu_1q, \ldots, pu_rq\} \subseteq B^{(n-1)}$ and using the induction hypothesis we get $\{pu_1q, \ldots, pu_rq\} \cdot C \subseteq (BC)^{(n-1)}$. So, $pu_1qc, \ldots, pu_rqc \in (BC)^{(n-1)}$. It follows that $pu_1 \ldots u_rqc = ac \in (BC)^{(n)}$. ∎

Proof of the theorem. Let $A, B, C \in \mathsf{F}(S)$, $A \varrho B$. We have to prove that $AC \varrho BC$, $CA \varrho CB$, $A \cup C \varrho B \cup C$.

(a) $AC \varrho BC$: We know that $[A] = [B]$. By 3.2 (iii), $[B] = \bigcup_{n=0}^{\infty} B^{(n)}$. So, $A \subseteq \bigcup_{n=0}^{\infty} B^{(n)}$. Since the set A is finite and in view of 3.2 (i), there is n_0 such that $A \subseteq B^{(n_0)}$. Using 3.3 we get $AC \subseteq (BC)^{(n_0)}$. But $[BC] = \bigcup_{n=0}^{\infty}(BC)^{(n)}$ (see 3.2.(iii)) which now implies $AC \subseteq [BC]$ and $[AC] \subseteq [BC]$. It can be proved similarly that $[BC] \subseteq [AC]$. So, $[AC] = [BC]$ and $AC \varrho BC$.

(b) $CA \varrho CB$: This case is similar to (a).

(c) $A \cup C \varrho B \cup C$: As in (a), $A \subseteq B^{(n_0)}$. It follows from 3.2 (ii) that $B^{(n_0)} \subseteq (B \cup C)^{(n_0)}$ and from 3.2 (i), 3.2 (ii) that $C = C^{(0)} \subseteq C^{(n_0)} \subseteq (B \cup C)^{(n_0)}$. Consequently, $B^{(n_0)} \cup C \subseteq (B \cup C)^{(n_0)}$. Now, $A \subseteq B^{(n_0)}$ implies $A \cup C \subseteq B^{(n_0)} \cup C$, thus $A \cup C \subseteq (B \cup C)^{(n_0)}$, $A \cup C \subseteq [B \cup C]$ (we have used 3.2 (iii)), $[A \cup C] \subseteq [B \cup C]$. Similarly, $[B \cup C] \subseteq [A \cup C]$. So, $[A \cup C] = [B \cup C], A \cup C \varrho B \cup C$.

It remains to prove that $\mathsf{F}(S)/\varrho \in \mathcal{SLOS}(x^r = x)$. Let $A \in \mathsf{F}(S)$. We want to show that $A^r \ \varrho \ A$. If $a \in A$ then $a = a^r \in A^r$. We see that $A \subseteq A^r, [A] \subseteq [A^r]$. If $x \in A^r$ then $x = a_1 \dots a_r$ for some $a_1, \dots, a_r \in A$. Since $1a_1 1, \dots, 1a_r 1 \in A = A^{(0)}$, we have $x = 1a_1 \dots a_r 1 \in A^{(1)}$. We see that $A^r \subseteq A^{(1)}$, $A^r \subseteq [A]$ (we have used 3.2 (iii)), $[A^r] \subseteq [A]$. Finally, $[A^r] = [A]$ and $A^r \ \varrho \ A$. ∎

Lemma 3.4. *Let $(S, \cdot) \in \mathcal{S}(x^r = x)$, $(A, \cdot, \vee) \in \mathcal{SLOS}(x^r = x)$, and let φ: $(S, \cdot) \to (A, \cdot)$ be a homomorphism. For any natural number n and any $B, C \in \mathsf{F}(S)$, if $B \subseteq C^{(n)}$ then $\bigvee_{b \in B} \varphi(b) \leq \bigvee_{c \in C} \varphi(c)$.*

Proof. By induction with respect to n.

$n = 0$: Clearly, if $B \subseteq C$ then $\bigvee_{b \in B} \varphi(b) \leq \bigvee_{c \in C} \varphi(c)$.

$n \geq 1$: Let $B \subseteq C^{(n)}$. Choose an arbitrary element $b \in B$. We want to show that $\varphi(b) \leq \bigvee_{c \in C} \varphi(c)$. Since $b \in C^{(n)}$, we have $b = pu_1 \dots u_r q$ for some $p, q \in S^1, u_1, \dots, u_r \in S$, with $pu_1 q, \dots, pu_r q \in C^{(n-1)}$. So, $\{pu_1 q, \dots, pu_r q\} \subseteq C^{(n-1)}$. Using the induction hypothesis we get $\varphi(pu_1 q) \vee \dots \vee \varphi(pu_r q) \leq \bigvee_{c \in C} \varphi(c)$. Now,

$$\begin{aligned}
\varphi(pu_1 q) \vee \dots \vee \varphi(pu_r q) &= \varphi(p)\varphi(u_1)\varphi(q) \vee \dots \vee \varphi(p)\varphi(u_r)\varphi(q) \\
&= \varphi(p)(\varphi(u_1) \vee \dots \vee \varphi(u_r))\varphi(q) \\
&= \varphi(p)(\varphi(u_1) \vee \dots \vee \varphi(u_r))^r \varphi(q) \\
&= \varphi(p)\Big(\bigvee_{i_1, \dots, i_r \in \{1,2,\dots,r\}} \varphi(u_{i_1})\varphi(u_{i_2}) \dots \varphi(u_{i_r})\Big)\varphi(q) \\
&= \bigvee_{i_1, \dots, i_r \in \{1,2,\dots,r\}} \varphi(p)\varphi(u_{i_1})\varphi(u_{i_2}) \dots \varphi(u_{i_r})\varphi(q) \\
&\geq \varphi(p)\varphi(u_1)\varphi(u_2) \dots \varphi(u_r)\varphi(q) \\
&= \varphi(pu_1 u_2 \dots u_r q) = \varphi(b)
\end{aligned}$$

So, $\varphi(b) \leq \bigvee_{c \in C} \varphi(c)$. The argument works here for $p, q \in S$ (the values $\varphi(p), \varphi(q)$ are defined). The case $p \notin S$ or $q \notin S$ is similar. ∎

Now, we are ready to present a construction of free objects in $\mathcal{SLOS}(x^r = x)$ using free objects in $\mathcal{S}(x^r = x)$.

Theorem 3.5. *Let S_r be a free object in $\mathcal{S}(x^r = x)$ on a set X with respect to a mapping ι: $X \to S_r$. Then $\mathsf{F}(S_r)/\varrho$ is a free object in $\mathcal{SLOS}(x^r = x)$ on X with respect to*

$$\kappa\colon\ X \to \mathsf{F}(S_r), \quad x \mapsto \{\iota(x)\}\varrho.$$

Proof. Write here S for S_r. Let $(A, \cdot, \vee) \in \mathcal{SLOS}(x^r = x)$, λ: $X \to A$. Clearly, $(A, \cdot) \in \mathcal{S}(x^r = x)$ and there is a unique homomorphism φ: $S \to (A, \cdot)$ satisfying $\varphi \circ \iota = \lambda$. Define a mapping ψ: $\mathsf{F}(S)/\varrho \to A$ in the following way:

for any $B \in \mathsf{F}(S)$,

$$\psi(B\varrho) = \bigvee_{b \in B} \varphi(b).$$

At first we prove the correctness of this definition. Let $B, C \in \mathsf{F}(S)$, $B \ \varrho \ C$. We want to show that $\bigvee_{b \in B} \varphi(b) = \bigvee_{c \in C} \varphi(c)$. We know that $[B] = [C] = \bigcup_{n=0}^{\infty} C^{(n)}$ (see 3.2 (iii)). In view of 3.2 (i), there is n_0 such that $B \subseteq C^{(n_0)}$. Using 3.4 we obtain $\bigvee_{b \in B} \varphi(b) \leq \bigvee_{c \in C} \varphi(c)$. On the other hand it can be proved similarly that $\bigvee_{c \in C} \varphi(c) \leq \bigvee_{b \in B} \varphi(b)$. Thus $\bigvee_{b \in B} \varphi(b) = \bigvee_{c \in C} \varphi(c)$.

The mapping ψ is a homomorphism from $\mathsf{F}(S)/\varrho$ to $(A, \cdot, \vee)$ satisfying $\psi \circ \kappa = \lambda$:

(a) Let $B, C \in \mathsf{F}(S)$. Then

$$\begin{aligned} \psi((B\varrho)(C\varrho)) &= \psi((BC)\varrho) = \bigvee_{b \in B, c \in C} \varphi(bc) = \bigvee_{b \in B, c \in C} \varphi(b)\varphi(c) \\ &= (\bigvee_{b \in B} \varphi(b)) \cdot (\bigvee_{c \in C} \varphi(c)) = \psi(B\varrho) \cdot \psi(C\varrho). \end{aligned}$$

(b) Let $B, C \in \mathsf{F}(S)$. Then

$$\begin{aligned} \psi((B\varrho) \vee (C\varrho)) &= \psi((B \cup C)\varrho) = \bigvee_{d \in B \cup C} \varphi(d) \\ &= (\bigvee_{b \in B} \varphi(b)) \vee (\bigvee_{c \in C} \varphi(c)) = \psi(B\varrho) \vee \psi(C\varrho). \end{aligned}$$

(c) Let $x \in X$. Then $\psi(\kappa(x)) = \psi(\{\iota(x)\}\varrho) = \varphi(\iota(x)) = \lambda(x)$.

Finally, let θ: $\mathsf{F}(S)/\varrho \to (A, \cdot, \vee)$ be a homomorphism with the property $\theta \circ \kappa = \lambda$. We have to prove that $\theta = \psi$. Define a mapping ϑ: $S \to A$ in this way: for any $s \in S$,

$$\vartheta(s) = \theta(\{s\}\varrho).$$

It is easy to check that ϑ is a homomorphism from S to $(A, \cdot)$ satisfying $\vartheta \circ \iota = \lambda$. Then $\vartheta = \varphi$. Now, for any $B \in \mathsf{F}(S)$, $\theta(B\varrho) = \theta((\bigcup_{b \in B}\{b\})\varrho) = \theta(\bigvee_{b \in B}(\{b\}\varrho)) = \bigvee_{b \in B} \theta(\{b\}\varrho) = \bigvee_{b \in B} \vartheta(b) = \bigvee_{b \in B} \varphi(b) = \psi(B\varrho)$ and thus $\theta = \psi$. ∎

Example (see also Romanowska [9], Remark 6.2). As an application, we will show a free semilattice-ordered band on the set $\{x, y\}$. Denote by S the free band on the set $\{x, y\}$. We know that S has six elements,

$$S = \{x, y, xy, yx, xyx, yxy\}.$$

Denote by T the set of all non-empty 2-closed subsets in S and define binary operations $\vee$ and $*$ on T by

$$A \vee B = [A \cup B], \quad A * B = [A \cdot B] \text{ for } A, B \in T.$$

Then $(T, *, \vee)$ is a free semilattice-ordered band on the set $\{x, y\}$. One can calculate that T has twenty elements:
$A_1 = \{x\}$, $A_2 = \{y\}$, $A_3 = \{xy\}$, $A_4 = \{yx\}$, $A_5 = \{xyx\}$, $A_6 = \{yxy\}$, $A_7 = \{x, xyx\}$, $A_8 = \{y, yxy\}$, $A_9 = \{xy, xyx\}$, $A_{10} = \{xy, yxy\}$, $A_{11} = \{yx, xyx\}$, $A_{12} = \{yx, yxy\}$, $A_{13} = \{x, xy, xyx\}$, $A_{14} = \{x, yx, xyx\}$, $A_{15} = \{y, xy, yxy\}$, $A_{16} = \{y, yx, yxy\}$, $A_{17} = \{xy, yx, xyx, yxy\}$, $A_{18} = \{x, xy, yx, xyx, yxy\}$, $A_{19} = \{y, xy, yx, xyx, yxy\}$, and $A_{20} = \{x, y, xy, yx,\ xyx, yxy\}$.

Since $A \leq B$ in the semilattice $(T, \vee)$ if and only if $A \subseteq B$, the order structure of $(T, \vee)$ is easily to imagine.

Recall that ρ is a *fully invariant* congruence on a semigroup $(S, \cdot)$ if it is a congruence on $(S, \cdot)$ such that for each $\alpha \in \mathsf{End}(S, \cdot)$ and $a, b \in S$ with $a\rho b$ we have also $\alpha(a)\rho\,\alpha(b)$. Let $\mathsf{Fic}(S, \cdot)$ denote the set of all fully invariant congruences on $(S, \cdot)$.

Denote by ρ_r the fully invariant congruence on X^+ corresponding to the variety $\mathcal{S}(x^r = x)$. Note that the relation ρ_r is expressed in the group-theoretical language in [4]. We will use the symbol $[\ ]_r$ for the closure operator $[\ ]$ on the semigroup X^+/ρ_r. We know that $X^\square = \mathsf{F}(X^+)$ is a free semilattice-ordered semigroup on the set X. Let $\sim_r$ be the fully invariant congruence on $X^\square$ corresponding to the variety $\mathcal{SLOS}(x^r = x)$. For any $Q \in X^\square$, let $Q\rho_r$ denote the set $\{q\rho_r \mid q \in Q\}$.

Corollary 3.6. *For any $Q, R \in X^\square$, we have $Q \sim_r R \Longleftrightarrow [Q\rho_r]_r = [R\rho_r]_r$.*

Proof. Clearly, $X^\square/\sim_r$ together with $\iota\colon x \to \{x\} \sim_r$, is a free object in $\mathcal{SLOS}(x^r = x)$ on X. Further, by 3.5., $\mathsf{F}(X^+/\rho_r)/\varrho$, with $\kappa\colon x \to \{x\rho_r\}\varrho$, is also a free object in $\mathcal{SLOS}(x^r = x)$ on X. So, there are mutually inverse isomorphisms

$$\varphi\colon X^\square/\sim_r \to \mathsf{F}(X^+/\rho_r)/\varrho \quad \text{and} \quad \psi\colon \mathsf{F}(X^+/\rho_r)/\varrho \to X^\square/\sim_r$$

such that $\varphi \circ \iota = \kappa, \psi \circ \kappa = \iota$.

Let $q = x_{i_1} \ldots x_{i_k} \in X^+$. Then

$$\begin{aligned}
\varphi(\{q\}\sim_r) &= \varphi(\{x_{i_1}\} \ldots \{x_{i_k}\}\sim_r) = \varphi((\{x_{i_1}\}\sim_r) \ldots (\{x_{i_k}\}\sim_r)) \\
&= \varphi(\{x_{i_1}\}\sim_r) \ldots \varphi(\{x_{i_k}\}\sim_r) = \varphi(\iota(x_{i_1})) \ldots \varphi(\iota(x_{i_k})) \\
&= \kappa(x_{i_1}) \ldots \kappa(x_{i_k}) = (\{x_{i_1}\rho_r\}\varrho) \ldots (\{x_{i_k}\rho_r\}\varrho) \\
&= \{x_{i_1}\rho_r\} \ldots \{x_{i_k}\rho_r\}\varrho = \{x_{i_1} \ldots x_{i_k}\rho_r\}\varrho = \{q\rho_r\}\varrho.
\end{aligned}$$

Now, let $Q \in X^\square$. Then

$$\begin{aligned}
\varphi(Q \sim_r) &= \varphi((\bigcup_{q \in Q} \{q\}) \sim_r) = \varphi(\bigvee_{q \in Q} (\{q\} \sim_r)) = \bigvee_{q \in Q} \varphi(\{q\} \sim_r) \\
&= \bigvee_{q \in Q} \{q\rho_r\}\varrho = (\bigcup_{q \in Q} \{q\rho_r\})\varrho = (Q\rho_r)\varrho.
\end{aligned}$$

Suppose that $Q, R \in X^{\square}$, $Q \sim_r R$. Consequently,

$$\varphi(Q{\sim_r}) = \varphi(R{\sim_r}),\ (Q\rho_r)\varrho = (R\rho_r)\varrho,\ (Q\rho_r)\ \varrho\ (R\rho_r),\ [Q\rho_r]_r = [R\rho_r]_r.$$

Finally, suppose that $Q, R \in X^{\square}$, $[Q\rho_r]_r = [R\rho_r]_r$. Then $(Q\rho_r)\varrho = (R\rho_r)\varrho$, $\varphi(Q \sim_r) = \varphi(R \sim_r)$. Using ψ we obtain $Q \sim_r R$. ■

4. Admissible closure operators

Let $X = \{x_1, x_2, \ldots\}$, $X_n = \{x_1, \ldots, x_n\}$ for $n = 1, 2, \ldots$ and use the following notation:

For any $\rho \in \mathsf{Fic}\, X^+$, $V \subseteq X^+/\rho$, $v \in X^+/\rho$, $q, p_1, p_2, \ldots \in X^+$, $x, y \in X$, $Q \subseteq X^+$ we define

$q(p_1, p_2, \ldots)$ as the word which results from q by substituting p_j for all occurrences of x_j, $j = 1, 2, \ldots$,

$v(p_1, p_2, \ldots) = (q(p_1, p_2, \ldots)\ )\,\rho$ where $q \in X^+$, $q\rho = v$,

$V(p_1, p_2, \ldots) = \{v(p_1, p_2, \ldots) \mid v \in V\}$,

$q(x \mid y)$ as the set of all words obtained from q by substituting y for **some** occurrences of x,

$Q\rho = \{q\rho \mid q \in Q\}$.

Now, we are going to introduce the key notion for us. Let $\rho \in \mathsf{Fic}\, X^+$. A mapping $[\ ]$: $\mathsf{P}(X^+/\rho) \to \mathsf{P}(X^+/\rho)$ will be called a *ρ-admissible closure operator* (or only an admissible closure operator if ρ is known from the context) if it satisfies the following conditions:

(I) $[\emptyset] = \emptyset$,

(II) $T \subseteq [T]$,

(III) $T \subseteq U$ implies $[T] \subseteq [U]$,

(IV) $[[T]] \subseteq [T]$,

(V) $[T] \subseteq \bigcup \{[V] \mid V$ is a finite subset of $T\}$,

(VI) $[\{t\}] = [\{u\}]$ implies $t = u$,

(VII) $[T]u \subseteq [Tu]$,

(VIII) $u[T] \subseteq [uT]$,

(IX) $[T](p_1, p_2, \ldots) \subseteq [T(p_1, p_2, \ldots)]$,

(X) $q\rho \in [\{q_1\rho, \ldots, q_k\rho\}]$ implies $(q(x \mid y))\rho \subseteq [(q_1(x \mid y))\rho \cup \cdots \cup (q_k(x \mid y))\rho]$, for any $T, U \in \mathsf{P}(X^+/\rho)$, $t, u \in X^+/\rho$, $q, q_1, \dots, q_k, p_1, p_2, \dots \in X^+$, $x, y \in X$.

Before formulating the main theorem we shall introduce some notations and prove a few lemmas.

For any natural numbers n, m, $i_1 < i_2 < \cdots < i_m$, any $q \in X^+$, $Q, P_1, P_2, \dots \in X^\square$, $x \in X$, we define

$\mathsf{c}(q)$ as the set of all variables from X which occur in q,

$\mathsf{c}(Q) = \bigcup_{q \in Q} \mathsf{c}(q)$,

$\mathsf{c}(x, q)$ as the number of occurrences of the variable x in q,

$X_n^\square = \{Q \in X^\square \mid \mathsf{c}(Q) \subseteq X_n \}$,

$Q(P_1, P_2, \dots)$ as the image of Q under the unique endomorphism on $X^\square$ which maps $\{x_j\}$ to P_j, $j = 1, 2, \dots$.

$Q(x_{i_1}/P_1, \dots, x_{i_m}/P_m)$ as the image of Q under the unique endomorphism on $X^\square$ which maps $\{x_{i_1}\}$ to $P_1, \dots, \{x_{i_m}\}$ to P_m and $\{x_j\}$ to $\{x_j\}$ for $j \notin \{i_1, \dots, i_m\}$.

Lemma 4.1. *For any $Q \in X^\square$, $p_1, p_2, \dots \in X^+$, we have*

$$Q(\{p_1\}, \{p_2\}, \dots) = \{q(p_1, p_2, \dots) \mid q \in Q\}.$$

Proof. Since

$$Q(\{p_1\}, \{p_2\}, \dots) = \left(\bigcup_{q \in Q} \{q\} \right) (\{p_1\}, \{p_2\}, \dots) = \bigcup_{q \in Q} \{q\}(\{p_1\}, \{p_2\}, \dots),$$

it is enough to show that $\{q\}(\{p_1\}, \{p_2\}, \dots) = \{q(p_1, p_2, \dots)\}$ for any $q \in X^+$. Let $q = x_{i_1} x_{i_2} \dots x_{i_k}$. Then

$$\begin{aligned} \{q\}(\{p_1\}, \{p_2\}, \dots) &= \{x_{i_1} x_{i_2} \dots x_{i_k}\}(\{p_1\}, \{p_2\}, \dots) \\ &= \{x_{i_1}\}\{x_{i_2}\} \dots \{x_{i_k}\}(\{p_1\}, \{p_2\}, \dots) \\ &= \{p_{i_1}\}\{p_{i_2}\} \dots \{p_{i_k}\} = \{p_{i_1} p_{i_2} \dots p_{i_k}\} = \{q(p_1, p_2, \dots)\}. \end{aligned}$$

Lemma 4.2. *For any $Q \in X^\square$, $x, y \in X$, we have $Q(x/\{x, y\}) = \bigcup_{q \in Q} q(x \mid y)$.*

Proof. It suffices to show that $\{q\}(x/\{x, y\}) = q(x \mid y)$ for any $q \in X^+$. The case $x = y$ is clear. So, let $x \neq y$. We will use the induction with respect to $\mathsf{c}(x, q)$.

In the case $\mathsf{c}(x, q) = 0$ the equality clearly holds.

So let $\mathsf{c}(x,q) \geq 1$, $q = rxs$, where $\mathsf{c}(x,s) = \mathsf{c}(x,q) - 1$, for some $r,s \in X^*$. Then

$$\begin{aligned}
\{q\}(x/\{x,y\}) &= \{r\}\{x\}\{s\}(x/\{x,y\}) \\
&= (\{r\}(x/\{x,y\}))(\{x\}(x/\{x,y\}))(\{s\}(x/\{x,y\})) \\
&= \{r\}\{x,y\}s(x \mid y) = \{rx,ry\}s(x \mid y) = (rxs)(x \mid y) = q(x \mid y).
\end{aligned}$$

Lemma 4.3. *For any natural numbers n, $i_1 < i_2 < \cdots i_n$, any $x \in X$, $x \notin \{x_{i_1},\ldots,x_{i_n}\}$, and any $Q \in X^{\Box}$ the following holds:*

$$Q(x/\{x_{i_1},\ldots,x_{i_n}\}) = ((\ldots(Q(x/\{x,x_{i_1}\}))\ldots)(x/\{x,x_{i_n}\}))(x/\{x_{i_1}\}).$$

Proof. It is enough to show that

$$\{y\}(x/\{x_{i_1},\ldots,x_{i_n}\}) = ((\ldots(\{y\}(x/\{x,x_{i_1}\}))\ldots)(x/\{x,x_{i_n}\}))(x/\{x_{i_1}\})$$

for any $y \in X$. The case $y \neq x$ is clear. If $y = x$ then $\{x\}(x/\{x_{i_1},\ldots,x_{i_n}\}) = \{x_{i_1},\ldots,x_{i_n}\}$ and

$$\begin{aligned}
&((\ldots(\{x\}(x/\{x,x_{i_1}\}))\ldots)(x/\{x,x_{i_n}\}))(x/\{x_{i_1}\}) \\
&\quad = ((\ldots(\{x,x_{i_1}\}(x/\{x,x_{i_2}\}))\ldots)(x/\{x,x_{i_n}\}))(x/\{x_{i_1}\}) \\
&\quad = ((\ldots(\{x,x_{i_1},x_{i_2}\}(x/\{x,x_{i_3}\}))\ldots)(x/\{x,x_{i_n}\}))(x/\{x_{i_1}\}) \\
&\quad = \{x,x_{i_1},x_{i_2},\ldots,x_{i_n}\}(x/\{x_{i_1}\}) = \{x_{i_1},\ldots,x_{i_n}\}.
\end{aligned}$$

Lemma 4.4. *Let $Q,P \in X^{\Box}$, $x \in X$. If $P = \{p_1,\ldots,p_n\}$, $i_1 < i_2 < \cdots < i_n$ are natural numbers, $x_{i_1},\ldots,x_{i_n} \notin \mathsf{c}(Q)$, then*

$$Q(x/P) = (Q(x/\{x_{i_1},\ldots,x_{i_n}\}))(x_{i_1}/\{p_1\},\ldots,x_{i_n}/\{p_n\}).$$

Proof. It suffices to show that

$$\{y\}(x/P) = (\{y\}(x/\{x_{i_1},\ldots,x_{i_n}\}))(x_{i_1}/\{p_1\},\ldots,x_{i_n}/\{p_n\})$$

for any $y \in X$ with $y \notin \{x_{i_1},\ldots,x_{i_n}\}$. So, let $y \in X$, $y \notin \{x_{i_1},\ldots,x_{i_n}\}$. If $y \neq x$, equality clearly holds. If $y = x$ then $\{x\}(x/P) = P$ and

$$\begin{aligned}
&(\{x\}(x/\{x_{i_1},\ldots,x_{i_n}\}))(x_{i_1}/\{p_1\},\ldots,x_{i_n}/\{p_n\}) \\
&\quad = \{x_{i_1},\ldots,x_{i_n}\}(x_{i_1}/\{p_1\},\ldots,x_{i_n}/\{p_n\}) = \{p_1,\ldots,p_n\} = P. \qquad \blacksquare
\end{aligned}$$

Lemma 4.5. *Let n be a natural number, $Q \in X^{\Box}_n$, $P_1,P_2,\ldots \in X^{\Box}$. If $i_1 < i_2 < \cdots < i_n$ are natural numbers, $\{x_{i_1},\ldots,x_{i_n}\} \cap (\mathsf{c}(P_1) \cup \cdots \cup \mathsf{c}(P_n)) = \emptyset$, then*

$$Q(P_1,P_2,\ldots) = (\ldots((Q(x_1/\{x_{i_1}\},\ldots,x_n/\{x_{i_n}\}))(x_{i_1}/P_1))\ldots)(x_{i_n}/P_n).$$

Proof. Obviously,

$$Q(P_1, P_2, \ldots) = (Q(x_1/\{x_{i_1}\}, \ldots, x_n/\{x_{i_n}\}))(x_{i_1}/P_1, \ldots, x_{i_n}/P_n).$$

Since $\{x_{i_1}, \ldots, x_{i_n}\} \cap (\mathsf{c}(P_1) \cup \cdots \cup \mathsf{c}(P_n)) = \emptyset$, $R(x_{i_1}/P_1, \ldots, x_{i_n}/P_n) = (\ldots R(x_{i_1}/P_1)\ldots)(x_{i_n}/P_n)$ for any $R \in X^\Box$. ∎

Let $\sim \in \mathsf{Fic}\, X^\Box$. We define a binary relation $\rho_\sim$ on X^+ in the following way: for any $q, r \in X^+$,

$$q \ \rho_\sim \ r \Longleftrightarrow \{q\} \sim \{r\}.$$

Obviously, $\rho_\sim$ is a congruence on X^+. Further, we define an operator $[\ ]_\sim \colon \mathsf{P}(X^+/\rho_\sim) \to \mathsf{P}(X^+/\rho_\sim)$ in such a way: for any $T \in \mathsf{P}(X^+/\rho_\sim)$, $q \in X^+$,

$$\begin{aligned} q\rho_\sim \in [T]_\sim \iff & \{q_1, \ldots, q_k, q\} \sim \{q_1, \ldots, q_k\} \\ & \text{for some natural number } k \text{ and } q_1, \ldots, q_k \in X^+ \\ & \text{such that } q_1\rho_\sim, \ldots, q_k\rho_\sim \in T. \end{aligned}$$

Note that the definition of the operator $[\ ]_\sim$ is correct. Let $q, r, q_1, \ldots, q_k \in X^+$, $q \ \rho_\sim \ r$, $\{q_1, \ldots, q_k, q\} \sim \{q_1, \ldots, q_k\}$, $q_1\rho_\sim, \ldots, q_k\rho_\sim \in T$. Since $\{q\} \sim \{r\}$, we get $\{q_1, \ldots, q_k, q\} \sim \{q_1, \ldots, q_k, r\}$ and so $\{q_1, \ldots, q_k, r\} \sim \{q_1, \ldots, q_k\}$.

Lemma 4.6. *Let* $\sim \in \mathsf{Fic}\, X^\Box$, $q, q_1, \ldots, q_k \in X^+$. *Then* $q\rho_\sim \in [\{q_1\rho_\sim, \ldots, q_k\rho_\sim\}]_\sim$ *if and only if* $\{q_1, \ldots, q_k, q\} \sim \{q_1, \ldots, q_k\}$.

Proof. Let $q\rho_\sim \in [\{q_1\rho_\sim, \ldots, q_k\rho_\sim\}]_\sim$. Then $\{r_1, \ldots, r_l, q\} \sim \{r_1, \ldots, r_l\}$ for some $r_1, \ldots, r_l \in X^+$, $r_1\rho_\sim, \ldots, r_l\rho_\sim \in \{q_1\rho_\sim, \ldots, q_k\rho_\sim\}$. Choose an arbitrary $j \in \{1, \ldots, l\}$. There exists $i_j \in \{1, \ldots, k\}$ such that $\{r_j\} \sim \{q_{i_j}\}$. Then

$$\{r_1, \ldots, r_l\} \sim \{q_{i_1}, \ldots, q_{i_l}\}, \ \{r_1, \ldots, r_l, q\} \sim \{q_{i_1}, \ldots, q_{i_l}, q\}.$$

It follows that $\{q_{i_1}, \ldots, q_{i_l}, q\} \sim \{q_{i_1}, \ldots, q_{i_l}\}, \{q_1, \ldots, q_k, q\} \sim \{q_1, \ldots, q_k\}$. The opposite implication is clear. ∎

Now we can formulate and prove our main theorem.

Theorem 4.7. (i) *Let* $\sim \in \mathsf{Fic}\, X^\Box$. *Then* $\rho_\sim \in \mathsf{Fic}\, X^+$ *and* $[\ ]_\sim$ *is a* $\rho_\sim$*-admissible closure operator.*

(ii) *Let* $\rho \in \mathsf{Fic}\, X^+$ *and let* $[\ ]$ *be a* ρ*-admissible closure operator. Let us define a binary relation* $\sim_{\rho,[\ ]}$ *on* $X^\Box$ *in the following way: for any* $Q, R \in X^\Box$ *we put*

$$Q \sim_{\rho,[\ ]} R \iff [Q\rho] = [R\rho].$$

Then $\sim_{\rho,[\ ]} \in \mathsf{Fic}\, X^\Box$.

(iii) *For any* $\sim \in \mathsf{Fic}\, X^{\square}$ *it is the case that* $\sim = \sim_{\rho_\sim, [\ ]_\sim}$.

(iv) *For any* $\rho \in \mathsf{Fic}\, X^+$ *and any* ρ*-admissible closure operator* $[\ ]$ *it is the case that*

$$\rho = \rho_{\sim_{\rho,[\ ]}}, \quad [\ ] = [\ ]_{\sim_{\rho,[\ ]}}.$$

Proof. Part (i). Let $q, r, p_1, p_2, \ldots \in X^+$, $\{q\} \sim \{r\}$. By 4.1 we have $\{q(p_1, p_2, \ldots)\} \sim \{r(p_1, p_2, \ldots)\}$. Thus $\rho_\sim \in \mathsf{Fic}\, X^+$. Now, we will consecutively prove that $[\ ]_\sim$ satisfies the conditions (I) – (X).

The conditions (I),(II),(III) are clearly satisfied.

(IV): Let $q\rho_\sim \in [[T]_\sim]_\sim$. There are $q_1, \ldots, q_k \in X^+$ such that $\{q_1, \ldots, q_k, q\} \sim \{q_1, \ldots, q_k\}$, $q_1\rho_\sim, \ldots, q_k\rho_\sim \in [T]_\sim$. For any $i \in \{1, \ldots, k\}$ we have: $\{r^i_1, \ldots, r^i_{l_i}, q_i\} \sim \{r^i_1, \ldots, r^i_{l_i}\}$ for some $r^i_1, \ldots, r^i_{l_i} \in X^+, r^i_1\rho_\sim, \ldots, r^i_{l_i}\rho_\sim \in T$. It holds: $\{r^1_1, \ldots, r^k_{l_k}, q_1, \ldots, q_k\} \sim \{r^1_1, \ldots, r^k_{l_k}\}$, $\{r^1_1, \ldots, r^k_{l_k}, q_1, \ldots, q_k, q\} \sim \{r^1_1, \ldots, r^k_{l_k}, q_1, \ldots, q_k\}$, $\{r^1_1, \ldots, r^k_{l_k}, q_1, \ldots, q_k, q\} \sim \{r^1_1, \ldots, r^k_{l_k}, q\}$. Hence $\{r^1_1, \ldots, r^k_{l_k}, q\} \sim \{r^1_1, \ldots, r^k_{l_k}\}$, and so $q\rho_\sim \in [T]_\sim$.

(V) is clearly satisfied.

(VI): Let $[\{t\}]_\sim = [\{u\}]_\sim$. There are $q, r \in X^+$ such that $t = q\rho_\sim, u = r\rho_\sim$. Now, $q\rho_\sim \in [\{r\rho_\sim\}]_\sim$ gives $\{q, r\} \sim \{r\}$, and $r\rho_\sim \in [\{q\rho_\sim\}]_\sim$ gives $\{q, r\} \sim \{q\}$ (see 4.6). Thus $\{q\} \sim \{r\}$, $t = u$.

(VII): Let $q\rho_\sim \in [T]_\sim$, $u = r\rho_\sim$ for some $r \in X^+$. We want to show that $(q\rho_\sim)(r\rho_\sim) \in [Tu]_\sim$. We know that $\{q_1, \ldots, q_k, q\} \sim \{q_1, \ldots, q_k\}$ for some $q_1, \ldots, q_k \in X^+$ with
$q_1\rho_\sim, \ldots, q_k\rho_\sim \in T$. It holds: $(q_1 r)\rho_\sim, \ldots, (q_k r)\rho_\sim \in Tu$, $\{q_1 r, \ldots, q_k r, qr\} \sim \{q_1 r, \ldots, q_k r\}$. Thus $(q\rho_\sim)(r\rho_\sim) \in [Tu]_\sim$.

(VIII) is similar to (VII).

(IX): Let $q\rho_\sim \in [T]_\sim$. We want to show that $(q(p_1, p_2, \ldots))\rho_\sim \in [T(p_1, p_2, \ldots)]_\sim$. We know that $\{q_1, \ldots, q_k, q\} \sim \{q_1, \ldots, q_k\}$ for some $q_1, \ldots, q_k \in X^+$ with $q_1\rho_\sim, \ldots, q_k\rho_\sim \in T$. Clearly, $(q_1(p_1, p_2, \ldots))\rho_\sim, \ldots, (q_k(p_1, p_2, \ldots))\rho_\sim \in T(p_1, p_2, \ldots)$. Further, by 4.1,

$$\begin{aligned} &\{q_1(p_1, p_2, \ldots), \ldots, q_k(p_1, p_2, \ldots), q(p_1, p_2, \ldots)\} \\ &\sim \{q_1(p_1, p_2, \ldots), \ldots, q_k(p_1, p_2, \ldots)\}, \end{aligned}$$

and so $(q(p_1, p_2, \ldots))\rho_\sim \in [T(p_1, p_2, \ldots)]_\sim$.

(X): Let $q\rho_\sim \in [\{q_1\rho_\sim, \ldots, q_k\rho_\sim\}]_\sim$. We want to show that

$$(q(x \mid y))\rho_\sim \subseteq [(q_1(x \mid y))\rho_\sim \cup \cdots \cup (q_k(x \mid y))\rho_\sim]_\sim.$$

We know that $\{q_1, \ldots, q_k, q\} \sim \{q_1, \ldots, q_k\}$ (see 4.6). By 4.2,

$$q_1(x \mid y) \cup \cdots \cup q_k(x \mid y) \cup q(x \mid y) \sim q_1(x \mid y) \cup \cdots \cup q_k(x \mid y).$$

For any $r \in q(x \mid y)$ we have $q_1(x \mid y) \cup \cdots \cup q_k(x \mid y) \cup \{r\} \sim q_1(x \mid y) \cup \cdots \cup q_k(x \mid y)$. Hence

$$r\rho_\sim \in [(q_1(x \mid y))\rho_\sim \cup \cdots \cup (q_k(x \mid y))\rho_\sim]_\sim \text{ and so}$$

$$(q(x \mid y))\rho_\sim \;\subseteq\; [(q_1(x \mid y))\rho_\sim \cup \cdots \cup (q_k(x \mid y))\rho_\sim]_\sim.$$

Part (ii). Clearly, $\sim_{\rho,[\;]}$ is an equivalence relation on $X^\Box$. We will prove that $\sim_{\rho,[\;]}$ is a congruence. Let $Q, R, S \in X^\Box$, $[Q\rho] = [R\rho]$. We want to show that

(a) $[(Q \cup S)\rho] = [(R \cup S)\rho]$: Clearly, $Q\rho \subseteq [R\rho] \subseteq [(R \cup S)\rho]$, $S\rho \subseteq [(R \cup S)\rho]$, which gives $(Q \cup S)\rho \subseteq [(R \cup S)\rho]$ and thus $[(Q \cup S)\rho] \subseteq [(R \cup S)\rho]$. The opposite inclusion can be shown in a similar way.

(b) $[(QS)\rho] = [(RS)\rho]$: Let $q \in Q$, $s \in S$. By (VII), $[R\rho](s\rho) \subseteq [(R\{s\})\rho]$. Since $q\rho \in [R\rho]$, we have $(qs)\rho \in [(R\{s\})\rho] \subseteq [(RS)\rho]$, thus $(QS)\rho \subseteq [(RS)\rho]$, which gives $[(QS)\rho] \subseteq [(RS)\rho]$. The opposite inclusion can be shown in a similar way.

(c) $[(SQ)\rho] = [(SR)\rho]$: It can be proved similarly as (b). ■

Now we prove a few auxiliary statements.

Statement 1. Let $Q, R \in X^\Box$, $p_1, p_2, \ldots \in X^+$ and $[Q\rho] = [R\rho]$. Then

$$[(Q(\{p_1\}, \{p_2\}, \ldots))\rho] = [(R(\{p_1\}, \{p_2\}, \ldots))\rho].$$

Proof. By 4.1.,

$$\begin{aligned} Q(\{p_1\}, \{p_2\}, \ldots) &= \{q(p_1, p_2, \ldots) \mid q \in Q\}, \\ R(\{p_1\}, \{p_2\}, \ldots) &= \{r(p_1, p_2, \ldots) \mid r \in R\}. \end{aligned}$$

For any $q \in Q$, $q\rho \in [R\rho]$. By (IX), $(q(p_1, p_2, \ldots))\rho \in [(R(\{p_1\}, \{p_2\}, \ldots))\rho]$, and thus

$$(Q(\{p_1\}, \{p_2\}, \ldots))\rho \subseteq [(R(\{p_1\}, \{p_2\}, \ldots))\rho],$$

which gives

$$[(Q(\{p_1\}, \{p_2\}, \ldots))\rho] \subseteq [(R(\{p_1\}, \{p_2\}, \ldots))\rho].$$

The opposite inclusion can be shown in a similar way.

Statement 2. Let $Q, R \in X^\Box$, $x, y \in X$ and $[Q\rho] = [R\rho]$. Then $[(Q(x/\{x,y\}))\rho] = [(R(x/\{x,y\}))\rho]$. ■

Proof. By 4.2, $Q(x/\{x,y\}) = \bigcup_{q \in Q} q(x \mid y)$, $R(x/\{x,y\}) = \bigcup_{r \in R} r(x \mid y)$. For any $q \in Q$, $q\rho \in [R\rho]$. From (X) we have $(q(x \mid y))\rho \subseteq [(R(x/\{x,y\}))\rho]$, and thus

$$(Q(x/\{x,y\}))\rho \subseteq [(R(x/\{x,y\}))\rho] \text{ and } [(Q(x/\{x,y\}))\rho] \subseteq [(R(x/\{x,y\}))\rho].$$

The opposite inclusion can be shown in a similar way.

Statement 3. Let $Q, R, P \in X^\Box$, $x \in X$ and $[Q\rho] = [R\rho]$. Then $[(Q(x/P))\rho] = [(R(x/P))\rho]$. ■

Proof. It follows from 4.3, 4.4 and the statements 1 and 2.

It remains to prove that the congruence $\sim_{\rho,[\ ]}$ is fully invariant. Let $Q, R, P_1, P_2, \ldots \in X^{\square}$, $[Q\rho] = [R\rho]$. We want to show that $[(Q(P_1, P_2, \ldots))\rho] = [(R(P_1, P_2, \ldots))\rho]$. There is a natural number n such that $Q, R \in X_n^{\square}$. Further, there are natural numbers $i_1 < i_2 < \cdots < i_n$ with the property $\{x_{i_1}, \ldots, x_{i_n}\} \cap (\mathsf{c}(P_1) \cup \cdots \cup \mathsf{c}(P_n)) = \emptyset$. Using 4.5 and the statements 1 and 3 we get the desired equality.

Part (iii).

Let $Q, R \in X^{\square}$, $Q \sim R$. For any $q \in Q$, $R \cup \{q\} \sim R$ and so $q\rho_{\sim} \in [R\rho_{\sim}]_{\sim}$, which gives $Q\rho_{\sim} \subseteq [R\rho_{\sim}]_{\sim}$, $[Q\rho_{\sim}]_{\sim} \subseteq [R\rho_{\sim}]_{\sim}$. The opposite inclusion can be proved similarly, altogether we have $Q \sim_{\rho_{\sim},[\ ]_{\sim}} R$. Let $Q, R \in X^{\square}$, $[Q\rho_{\sim}]_{\sim} = [R\rho_{\sim}]_{\sim}$. For any $q \in Q$, $q\rho_{\sim} \in [R\rho_{\sim}]_{\sim}$. Therefore $R \cup \{q\} \sim R$. Hence $Q \cup R \sim R$. Similarly, we have $Q \cup R \sim Q$, which gives altogether $Q \sim R$.

Part (iv).

$\rho \subseteq \rho_{\sim_{\rho,[\ ]}}$: Let $q\ \rho\ r$. Then $[\{q\rho\}] = [\{r\rho\}], \{q\} \sim_{\rho,[\ ]} \{r\}$, which we need.

$\rho_{\sim_{\rho,[\ ]}} \subseteq \rho$: Let $[\{q\rho\}] = [\{r\rho\}]$. Then, by (VI), $q\ \rho\ r$.

We already know that $[\ ], [\ ]_{\sim_{\rho,[\ ]}}\colon \mathsf{P}(X^+/\rho) \to \mathsf{P}(X^+/\rho)$. Let $T \in \mathsf{P}(X^+/\rho)$.

$[T] \subseteq [T]_{\sim_{\rho,[\ ]}}$: Let $q \in X^+, q\rho \in [T]$. By (V), there is a finite subset V of T such that $q\rho \in [V]$. By (I), $V \neq \emptyset$ and so $V = \{q_1\rho, \ldots, q_k\rho\}$ for some $q_1, \ldots, q_k \in X^+$. It holds: $[\{q_1\rho, \ldots, q_k\rho, q\rho\}] = [\{q_1\rho, \ldots, q_k\rho\}]$. Thus $\{q_1, \ldots, q_k, q\} \sim_{\rho,[\ ]} \{q_1, \ldots, q_k\}$, which gives $q\rho \in [T]_{\sim_{\rho,[\ ]}}$.

$[T]_{\sim_{\rho,[\ ]}} \subseteq [T]$: Let $q \in X^+, q\rho \in [T]_{\sim_{\rho,[\ ]}}$. Then $[\{q_1\rho, \ldots, q_k\rho, q\rho\}] = [\{q_1\rho, \ldots, q_k\rho\}]$ for some $q_1, \ldots, q_k \in X^+$ with $q_1\rho, \ldots, q_k\rho \in T$. Hence $q\rho \in [\{q_1\rho, \ldots, q_k\rho\}] \subseteq [T]$.

The proof of our theorem is complete. ∎

Let $\mathcal{V}$ be a variety of semigroups determined by a set $\Sigma \subseteq X^+ \times X^+$ of identities. Denote by $\rho_{\mathcal{V}}$ the fully invariant congruence on X^+ corresponding to the variety $\mathcal{V}$. Let $\mathcal{SLOV}$ be the class of all semilattice-ordered semigroups $(A, \cdot, \vee)$ with the property $(A, \cdot) \in \mathcal{V}$. Clearly, $\mathcal{SLOV}$ is the variety determined by the set of identities $\{(\{u\}, \{v\}) \mid (u, v) \in \Sigma\}$. Let $\mathcal{W}$ be a variety of semilattice-ordered semigroups, $\sim_{\mathcal{W}}$ be the corresponding fully invariant congruence on $X^{\square}$. By 4.7, $\sim_{\mathcal{W}}$ corresponds to the ordered pair $(\rho, [\ ])$, where $\rho \in \mathsf{Fic}\, X^+$ and $[\ ]$ is a ρ-admissible closure operator (in the notation of 4.7, $\sim_{\mathcal{W}} = \sim_{\rho,[\ ]}$). Now,

$$\mathcal{W} \subseteq \mathcal{SLOV} \Leftrightarrow \sim_{\mathcal{W}} \supseteq \{(\{u\}, \{v\}) \mid (u, v) \in \Sigma\} \Leftrightarrow \rho \supseteq \Sigma \Leftrightarrow \rho \supseteq \rho_{\mathcal{V}}.$$

So, the theorem 4.7 reduces finding of all subvarieties in $\mathcal{SLOV}$ to the description of all ρ-admissible closure operators for all $\rho \in \mathsf{Fic}\, X^+$ such that $\rho \supseteq \rho_{\mathcal{V}}$.

Remark 4.8. Note that in the proof of the second part of 4.7 we used only the properties (II), (III), (IV), (VII), (VIII), (IX) and (X) of the operator $[\ ]$.

We will show that all these seven conditions are satisfied by the operator $[\ ]_2$ from the section 3. The conditions (II), (III) and (IV) are clearly satisfied. It can be easily proved by induction, using 3.2 (iii), that the operator $[\ ]_2$ also satisfies the conditions (VII), (VIII) and (IX). Now, we prove in details that the operator $[\ ]_2$ has the property (X). Write ρ, $[\ ]$ instead of ρ_2, $[\ ]_2$. At first, we have the following

Claim. Let $q, r \in X^+$, $x, y \in X$. If $q\ \rho\ r$ then $[(q(x \mid y))\rho] = [(r(x \mid y))\rho]$.

Proof. Let $c, e \in X^*, d \in X^+$. We will show that $[((cde)(x \mid y))\rho] = [((cdde)(x \mid y))\rho]$. Indeed, $[((cde)(x \mid y))\rho] \subseteq [((cdde)(x \mid y))\rho]$ is clear since $((cde)(x \mid y))\rho \subseteq ((cdde)(x \mid y))\rho$. To get the opposite inclusion $[((cdde)(x \mid y))\rho] \subseteq [((cde)(x \mid y))\rho]$, let $w \in (cdde)(x \mid y)$. Then $w = c_1d_1d_2e_1$ for some $c_1 \in c(x \mid y)$, $d_1, d_2 \in d(x \mid y)$, $e_1 \in e(x \mid y)$ (here, we put $1(x \mid y) = \{1\}$). Now, $(c_1d_1e_1)\rho, (c_1d_2e_1)\rho \in [((cde)(x \mid y))\rho]$. It follows that $(c_1d_1d_2e_1)\rho = w\rho \in [((cde)(x \mid y))\rho]$. We have proved that $((cdde)(x \mid y))\rho \subseteq [((cde)(x \mid y))\rho]$. Consequently, $[((cdde)(x \mid y))\rho] \subseteq [((cde)(x \mid y))\rho]$. The proof of our claim is complete.

We continue in proving that $[\ ]$ satisfies the condition (X).

By 3.2 (iii), $[\{q_1\rho, \ldots, q_k\rho\}] = \bigcup_{n=0}^{\infty}\{q_1\rho, \ldots, q_k\rho\}^{(n)}$. We will prove by induction that $q\rho \in \{q_1\rho, \ldots, q_k\rho\}^{(n)}$ implies $(q(x \mid y))\rho \subseteq [(q_1(x \mid y))\rho \cup \cdots \cup (q_k(x \mid y))\rho]$ (for $n = 0, 1, 2, \ldots$).

$n = 0$: Let $q\rho \in \{q_1\rho, \ldots, q_k\rho\}^{(0)}$. Then $q\ \rho\ q_i$ for some $i \in \{1, \ldots, k\}$. By the auxiliary statement, $[(q(x \mid y))\rho] = [(q_i(x \mid y))\rho]$ and so $(q(x \mid y))\rho \subseteq [(q_1(x \mid y))\rho \cup \cdots \cup (q_k(x \mid y))\rho]$.

$n \geq 1$: Let $q\rho \in \{q_1\rho, \ldots, q_k\rho\}^{(n)}$. There are $c, e \in X^*, d_1, d_2 \in X^+$, such that $(cd_1e)\rho$, $(cd_2e)\rho \in \{q_1\rho, \ldots, q_k\rho\}^{(n-1)}$, $q\ \rho\ cd_1d_2e$. By the induction hypothesis,

$$((cd_1e)(x \mid y))\rho \ \subseteq\ [(q_1(x \mid y))\rho \cup \cdots \cup (q_k(x \mid y))\rho],$$
$$((cd_2e)(x \mid y))\rho \ \subseteq\ [(q_1(x \mid y))\rho \cup \cdots \cup (q_k(x \mid y))\rho].$$

Let $w \in (cd_1d_2e)(x \mid y)$. Then $w = c'd_1'd_2'e'$ for some $c' \in c(x \mid y)$, $d_1' \in d_1(x \mid y)$, $d_2' \in d_2(x \mid y)$, $e' \in e(x \mid y)$. Now, $c'd_1'e' \in (cd_1e)(x \mid y)$, $c'd_2'e' \in (cd_2e)(x \mid y)$. It follows that

$$(c'd_1'e')\rho, (c'd_2'e')\rho \in [(q_1(x \mid y))\rho \cup \cdots \cup (q_k(x \mid y))\rho],$$

$$(c'd_1'd_2'e')\rho = w\rho \in [(q_1(x \mid y))\rho \cup \cdots \cup (q_k(x \mid y))\rho].$$

We have proved that $((cd_1d_2e)(x \mid y))\rho \subseteq [(q_1(x \mid y))\rho \cup \cdots \cup (q_k(x \mid y))\rho]$. Since $[(q(x \mid y))\rho] = [((cd_1d_2e)(x \mid y))\rho]$ (see the claim), we get $(q(x \mid y))\rho \subseteq [(q_1(x \mid y))\rho \cup \cdots \cup (q_k(x \mid y))\rho]$.

We have just completed our proof of the fact that the operator $[\ ]_2$ satisfies the conditions (II), (III), (IV), (VII), (VIII), (IX) and (X). Hence the pair $(\rho_2, [\ ]_2)$ (recall that ρ_2 denotes the fully invariant congruence on X^+

corresponding to the variety of all bands) determines a fully invariant congruence on $X^\square$. In fact, the congruence determined by $(\rho_2, [\,]_2)$ is $\sim_2$, the fully invariant congruence on $X^\square$ corresponding to the variety of all semilattice-ordered bands (see 3.6.). But, as shown by Michal Kunc (Brno) [5], $[\,]_2$ does not satisfy the condition (VI) and so $[\,]_2$ is not a ρ_2-admissible closure operator (see also the proof of Theorem 2.3 in [7]). Indeed,

$$(xy\rho_2)(x\rho_2)(yz\rho_2) = xyz\rho_2, \quad (xy\rho_2)(zx\rho_2)(yz\rho_2) = xyz\rho_2,$$

which implies $xyxzxyz\rho_2 \in [\{xyz\rho_2\}]_2$.

Further,

$$(xyx\rho_2)(y\rho_2)(xzxyz\rho_2) = xyxzxyz\rho_2, (xyx\rho_2)(z\rho_2)(xzxyz\rho_2) = xyxzxyz\rho_2,$$

which implies $xyz\rho_2 \in [\{xyxzxyz\rho_2\}]_2$. We see that $[\{xyz\rho_2\}]_2 = [\{xyxzxyz\rho_2\}]_2$ but $xyz\rho_2 \neq xyxzxyz\rho_2$ (here, x, y, z are three distinct elements from X). So, $(\rho_2, [\,]_2) \neq (\rho_{\sim_2}, [\,]_{\sim_2})$. In view of 2.9., $\rho_{\mathbf{RegB}} \subseteq \rho_{\sim_2}$, where $\rho_{\mathbf{RegB}}$ denotes the fully invariant congruence on X^+ corresponding to the variety of all regular bands. It can be proved that $\rho_{\mathbf{RegB}} = \rho_{\sim_2}$. ∎

The following result gives the construction of the intersection of fully invariant congruences on $X^\square$ in terms of pairs $(\rho, [\,])$.

Theorem 4.9. *Let $\rho_1, \rho_2 \in \mathsf{Fic}\, X^+$, $[\,]_1$ be a ρ_1-admissible closure operator, $[\,]_2$ be a ρ_2-admissible closure operator. Put $\sim\, = \,\sim_{\rho_1,[\,]_1} \cap \sim_{\rho_2,[\,]_2}$. Then*

$$\rho_\sim = \rho_1 \cap \rho_2,$$

$$\begin{aligned} q\rho_\sim &\in [\{q_1\rho_\sim, \dots, q_k\rho_\sim\}]_\sim \Longleftrightarrow q\rho_1 \in [\{q_1\rho_1, \dots, q_k\rho_1\}]_1, \\ q\rho_2 &\in [\{q_1\rho_2, \dots, q_k\rho_2\}]_2 \end{aligned}$$

$(q, q_1, \dots, q_k \in X^+)$.

Proof. Let $q, r \in X^+$. It holds:

$$\begin{aligned} (q,r) \in \rho_\sim &\Leftrightarrow (\{q\},\{r\}) \in \sim \Leftrightarrow (\{q\},\{r\}) \in \sim_{\rho_1,[\,]_1}, (\{q\},\{r\}) \in \sim_{\rho_2,[\,]_2} \\ &\Leftrightarrow (q,r) \in \rho_1, (q,r) \in \rho_2 \Leftrightarrow (q,r) \in \rho_1 \cap \rho_2. \end{aligned}$$

Further, for any $q, q_1, \dots, q_k \in X^+$,

$$\begin{aligned} q\rho_\sim \in [\{q_1\rho_\sim, \dots, q_k\rho_\sim\}]_\sim &\Leftrightarrow \{q_1, \dots, q_k, q\} \sim \{q_1, \dots, q_k\} \\ &\Leftrightarrow (\{q_1, \dots, q_k, q\}, \{q_1, \dots, q_k\}) \in \sim_{\rho_1,[\,]_1} \cap \sim_{\rho_2,[\,]_2} \\ &\Leftrightarrow q\rho_1 \in [\{q_1\rho_1, \dots, q_k\rho_1\}]_1, \end{aligned}$$

$q\rho_2 \in [\{q_1\rho_2, \dots, q_k\rho_2\}]_2$. ∎

The operator $[\,]_\sim$ from 4.9. will be denoted by $[\,]_1 \wedge [\,]_2$. In this notation,

$$\sim_{\rho_1,[\,]_1} \cap \sim_{\rho_2,[\,]_2} = \sim_{\rho_1\cap\rho_2,[\,]_1\wedge[\,]_2} .$$

Corollary 4.10. *Let $\rho_1, \rho_2 \in \mathsf{Fic}\, X^+$, $[\,]_1$ be a ρ_1-admissible closure operator, $[\,]_2$ be a ρ_2-admissible closure operator. Then $\sim_{\rho_1,[\,]_1} \subseteq \sim_{\rho_2,[\,]_2}$ if and only if $\rho_1 \subseteq \rho_2$ and, for any $q, q_1, \ldots, q_k \in X^+$, $q\rho_1 \in [\{q_1\rho_1, \ldots, q_k\rho_1\}]_1$ implies $q\rho_2 \in [\{q_1\rho_2, \ldots, q_k\rho_2\}]_2$.*

5. Admissible partial orders

Let $\rho \in \mathsf{Fic}\, X^+$. A binary relation $\leq$ on X^+/ρ will be called a *ρ-admissible partial order* (or only an admissible partial order if ρ is known from the context) if it satisfies the following conditions:

(A) $u \leq u$,

(B) $u \leq v, v \leq u$ implies $u = v$,

(C) $u \leq v, v \leq w$ implies $u \leq w$,

(D) $u \leq v$ implies $uw \leq vw$,

(E) $u \leq v$ implies $wu \leq wv$,

(F) $u \leq v$ implies $u(p_1, p_2, \ldots) \leq v(p_1, p_2, \ldots)$,

(G) $q\rho \leq r\rho$, $x \notin \mathsf{c}(r)$, $s \in q(x \mid y)$ implies $s\rho \leq r\rho$

for any $u, v, w \in X^+/\rho$, $q, r, s, p_1, p_2, \ldots \in X^+$, $x, y \in X$.

The reason for introducing this new notion follows from the next theorem. It will help us by finding ρ-admissible closure operators.

Theorem 5.1. *Let $\rho \in \mathsf{Fic}\, X^+$, $[\,]$ be a ρ-admissible closure operator. Let us define on X^+/ρ a binary relation $\leq$ in this way: for any $u, v \in X^+/\rho$,*

$$u \leq v \iff u \in [\{v\}].$$

Then the relation $\leq$ is a ρ-admissible partial order.

Proof. (A) is clear.

(B): Let $u \in [\{v\}]$, $v \in [\{u\}]$. We want to show that $u = v$. We see that $[\{u\}] = [\{v\}]$. The desired equality follows from (VI).

(C): Let $u \in [\{v\}]$, $v \in [\{w\}]$. We want to show that $u \in [\{w\}]$. We see that $[\{v\}] \subseteq [\{w\}]$ and thus $u \in [\{w\}]$.

(D) follows immediately from (VII).

(E) follows immediately from (VIII).

(F) follows immediately from (IX).

(G): Let $q\rho \in [\{r\rho\}]$, $x \notin \mathsf{c}(r)$, $s \in q(x \mid y)$. We want to show that $s\rho \in [\{r\rho\}]$. By (X), $(q(x \mid y))\rho \subseteq [(r(x \mid y))\rho]$. Since $x \notin \mathsf{c}(r)$, $r(x \mid y) = \{r\}$. Thus $(q(x \mid y))\rho \subseteq [\{r\rho\}]$ and $s\rho \in [\{r\rho\}]$. ∎

6. Applications

Let $\mathcal{SL}$ denote the class of all semilattices. We know that $q\ \rho_{\mathcal{SL}}\ r$ if and only if $\mathsf{c}(q) = \mathsf{c}(r)$ and thus $(X^+/\rho_{\mathcal{SL}}, \cdot)$ is isomorphic to $(\mathsf{F}(X), \cup)$, where $\mathsf{F}(X)$ is the set of all non-empty finite subsets of X. One can show that there are exactly three admissible partial orders on $\mathsf{F}(X)$ namely for $Y, Z \in \mathsf{F}(X)$,

$$Y \leq_1 Z \iff Y = Z, \quad Y \leq_2 Z \iff Y \subseteq Z \text{ and } Y \leq_3 Z \iff Y \supseteq Z$$

Moreover, there are exactly four $\rho_{\mathcal{SL}}$-admissible closure operators; namely $Y \in [\{Y_1, \dots, Y_k\}]_i$ if and only if

$$Y = Y_{j_1} \cup \dots \cup Y_{j_l} \text{ for some } j_1, \dots, j_l \in \{1, \dots, k\} \text{ in case } i = 1,$$

$$Y_j \subseteq Y \subseteq Y_1 \cup \dots \cup Y_k \text{ for some } j \in \{1, \dots, k\} \text{ in case } i = 2,$$

$$Y \subseteq Y_1 \cup \cdots Y_k \text{ in case } i = 3,$$

$$Y_j \subseteq Y \text{ for some } j \in \{1, \dots, k\} \text{ in case } i = 4.$$

The first two operators correspond to $\leq_1$, the third to $\leq_2$ and the last one to $\leq_3$. Thus considering also the trivial variety there are exactly five varieties of semilattice-ordered semigroups, which were found already by McKenzie and Romanowska in [11]. This together with analogous results for other varieties of normal bands leads to a complete descripton of all varieties of semilattice-ordered normal bands will be proved in a subsequent paper [6]. Note that the lattice of all varieties of semilattice-ordered normal bands was independently found also by Ghosh, Pastijn and Zhao [12] Theorem 4.9. Nevertheless, using our closure operators we can solve the word problem in each mentioned variety.

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Department of Mathematics
J.E.Purkyně University
České mládeže 8
400 96 Ústí nad Labem, Czech Republic
kurilm@pf.ujep.cz

Department of Mathematics
Masaryk University
Janáčkovo nám 2a
662 95 Brno, Czech Republic
polak@math.muni.cz
http://www.math.muni.cz/~polak

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Admissible closure operators and varieties of semilattice-ordered normal bands

Martin Kuřil

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Abstract. It is known that varieties of semilattice-ordered semigroups are in one-to-one correspondence with the ordered pairs $(\rho, [\ ])$ where ρ is a fully invariant congruence on the free semigroup on a countably infinite set and $[\ ]$ is a ρ-admissible closure operator. We find all admissible closure operators for varieties of left normal bands. Using the obtained results we describe all varieties of semilattice-ordered left normal bands by admissible closure operators. We solve the identity problem for all varieties of semilattice-ordered normal bands.

1. Introduction

A structure $(S, \cdot, \vee)$ is called a *semilattice-ordered semigroup* if
- (i) $(S, \cdot)$ is a semigroup,
- (ii) $(S, \vee)$ is a semilattice,
- (iii) $a(b \vee c) = ab \vee ac$, $(b \vee c)a = ba \vee ca$ for all $a, b, c \in S$.

Let X be a countably infinite set. Then X^+ denotes the free semigroup on X and $X^* = X^+ \cup \{\varepsilon\}$, where ε denotes the empty word. Further, the set of all fully invariant congruences on an algebraic structure A is denoted by $\mathsf{Fic}A$, the set of all endomorphisms of A is denoted by $\mathsf{End}A$, the set of all subsets of a set Y is denoted by $\mathsf{P}(Y)$ and the set of all finite non-empty subsets of Y is denoted by $\mathsf{P}_f(Y)$.

In [7], Polák and the author have introduced the notion of admissible closure operators. Let $\rho \in \mathsf{Fic}X^+$. A mapping $[\ ]\colon \mathsf{P}(X^+/\rho) \to \mathsf{P}(X^+/\rho)$ is called a ρ-*admissible closure operator* if it is a finitary closure operator which preserves the empty set and satisfies certain axioms (the axioms are presented in detail in the

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next section). It is proved in [7] that the varieties of semilattice-ordered semigroups are in one-to-one correspondence with the ordered pairs $(\rho, [\ ])$ where $\rho \in \mathsf{Fic} X^+$ and $[\ ]$ is a ρ-admissible closure operator.

We note that recently Pilitowska and Zamojska-Dzienio studied in [10] similar connections between closure operators on an arbitrary algebra A and congruences on the semilattice-ordered algebra of non-empty subsets of A. They also used these connections to give a description of the lattice of all subvarieties of semilattice-ordered algebras.

This article gives an application of results from [7]. Gajdoš and the author in [3] have already used these results for the construction of free semilattice-ordered semigroups satisfying $x^n = x$. Some other authors have also mentioned or applied the results from [7] – see [2], [10], [11], [12], [13], [14], [15], [16].

Let $\mathcal{V}$ be a variety of semigroups. Denote by $\rho_{\mathcal{V}}$ the fully invariant congruence on X^+ corresponding to the variety $\mathcal{V}$ and by $\boldsymbol{SLO}\mathcal{V}$ the class of all semilattice-ordered semigroups $(S, \cdot, \vee)$ with the property $(S, \cdot) \in \mathcal{V}$. It is easy to see that $\boldsymbol{SLO}\mathcal{V}$ is a variety.

Results 2.1 and 2.2 reduce finding all subvarieties in $\boldsymbol{SLO}\mathcal{V}$ to the description of all ρ-admissible closure operators for all $\rho \in \mathsf{Fic} X^+$ such that $\rho \supseteq \rho_{\mathcal{V}}$ (see Remark 2.3).

We denote by $\boldsymbol{B}$ ($\boldsymbol{T}$, $\boldsymbol{LZ}$, $\boldsymbol{RZ}$, $\boldsymbol{SL}$, $\boldsymbol{LNB}$, $\boldsymbol{RNB}$, $\boldsymbol{ReB}$, $\boldsymbol{NB}$) the variety of all bands (trivial semigroups, semigroups of left zeros, semigroups of right zeros, semilattices, left normal bands, right normal bands, rectangular bands, normal bands). Further, let $\mathcal{W}$ be a variety of (semilattice-ordered) semigroups. Then $\mathcal{L}(\mathcal{W})$ stands for the lattice of all subvarieties in $\mathcal{W}$.

In this article we find all admissible closure operators for varieties of left normal bands (see Theorems 4.2, 4.3, 4.4 and 4.6). Thus we obtain a complete description of the lattice $\mathcal{L}(\boldsymbol{SLOLNB})$ with all pairs $(\rho, [\ ])$ where $\rho \in \mathsf{Fic} X^+$, $\rho \supseteq \rho_{\boldsymbol{LNB}}$, $[\ ]$ is a ρ-admissible closure operator (see Theorem 5.1). Of course, in a dual way, we obtain a complete description of the lattice $\mathcal{L}(\boldsymbol{SLORNB})$ with all pairs $(\rho, [\ ])$ where $\rho \in \mathsf{Fic} X^+$, $\rho \supseteq \rho_{\boldsymbol{RNB}}$, $[\ ]$ is a ρ-admissible closure operator. We present varieties of semilattice-ordered left normal bands by admissible closure operators and simultaneously also by defining identities (see Remark 5.2) Finally, in Section 6, we present a solution of the identity problem for all varieties of semilattice-ordered normal bands.

A description of the lattice $\mathcal{L}(\boldsymbol{SLONB})$ has been published by Ghosh, Pastijn and Zhao in [4, Theorem 4.9]. Pastijn even described the lattice $\mathcal{L}(\boldsymbol{SLOB})$ in [8].

The lattice $\mathcal{L}(\boldsymbol{SLOLNB})$ is a sublattice of $\mathcal{L}(\boldsymbol{SLONB})$. This raises the natural question why Theorem 5.1 is presented in the paper. It is for two reasons:

We found all varieties of semilattice-ordered left normal bands using admissible closure operators only and we describe all the varieties with ordered pairs $(\rho, [\ ])$, where $\rho \in \{\rho_{\boldsymbol{T}}, \rho_{\boldsymbol{LZ}}, \rho_{\boldsymbol{SL}}, \rho_{\boldsymbol{LNB}}\}$ and $[\ ]$ is a ρ-admissible closure operator. This approach is different from that which was applied in the article [4].

All varieties of semilattice-ordered normal bands, obtained with the method used in this paper, have been presented by the author in his talk at the Colloquium on Semigroups in Szeged, 17 July, 2000. All ρ-admissible closure operators for $\rho \in \{\rho_{\boldsymbol{SL}}, \rho_{\boldsymbol{LZ}}, \rho_{\boldsymbol{RZ}}\}$ have already been found by the author in his master's thesis (see [6, Theorems 5.2., 5.3.]). There is also a conjecture in [6] that the lattice $\mathcal{L}(\boldsymbol{SLONB})$ consists of 35 varieties.

2. Admissible closure operators

In this section we recall the notion of admissible closure operator and the main result concerning it from [7]. It will play a central role in our paper.

We put, for any semigroup S and any $Q, R \in \mathsf{P}(S)$,

$$Q \cdot R = \{qr \mid q \in Q, r \in R\}.$$

Then $(\mathsf{P}_f(S), \cdot, \cup)$ is a semilattice-ordered semigroup. Here, as usual, $\cup$ denotes the set-theoretical union.

Result 2.1. ([7, Theorem 2.5]) *The structure* $(\mathsf{P}_f(X^+), \cdot, \cup)$ *together with the embedding* $x \mapsto \{x\}$, $x \in X$, *is a free object on the set* X *in the variety of all semilattice-ordered semigroups.*

The algebraic structure $(\mathsf{P}_f(X^+), \cdot, \cup)$ will be briefly denoted by $X^{\square}$.

As it is well known, varieties of semilattice-ordered semigroups are in bijective correspondence with fully invariant congruences on $X^{\square}$ (see [1, Proposition 8.6.4]).

Let $r \in X^*$. The set of all variables in r is denoted by $\mathsf{c}(r)$.

For any $q \in X^+$, $x, y \in X$, we define $q(x|y)$ as the set of all words obtained from q by substituting y for some occurrences of x. More formally: If $x \notin \mathsf{c}(q)$ then $q(x|y) = \{q\}$. If $x \in \mathsf{c}(q)$, $q = q_1 x q_2 x \dots q_{k-1} x q_k$, where $q_i \in X^*$, $x \notin \mathsf{c}(q_i)$ $(i = 1, 2, \dots, k)$, then

$$q(x|y) = \{q_1\}\{x, y\}\{q_2\}\{x, y\} \dots \{q_{k-1}\}\{x, y\}\{q_k\}.$$

We give a few examples. Let x, y, z be three different variables. We have

$$x^2(x|y) = \varepsilon x \varepsilon x \varepsilon(x|y) = \{\varepsilon\}\{x, y\}\{\varepsilon\}\{x, y\}\{\varepsilon\} = \{x^2, xy, yx, y^2\},$$
$$xz(x|y) = \varepsilon xz(x|y) = \{\varepsilon\}\{x, y\}\{z\} = \{xz, yz\},$$

$$xz(x|x) = \varepsilon xz(x|x) = \{\varepsilon\}\{x,x\}\{z\} = \{\varepsilon\}\{x\}\{z\} = \{xz\}.$$

Now, we are coming to the key notion. Let $\rho \in \mathsf{Fic}X^+$. A mapping $[\ ]\colon \mathsf{P}(X^+/\rho) \to \mathsf{P}(X^+/\rho)$ is called a *ρ-admissible closure operator* (or only an admissible closure operator if ρ is known from the context) if it is a finitary closure operator on X^+/ρ which preserves the empty set and satisfies the following conditions:

(I) $[\{t\}] = [\{u\}]$ implies $t = u$,
(II) $[T][U] \subseteq [TU]$,
(III) $f([T]) \subseteq [f(T)]$,
(IV) $q\rho \in [\{q_1\rho, \dots, q_k\rho\}]$ implies $q(x|y)\rho \subseteq [q_1(x|y)\rho \cup \dots \cup q_k(x|y)\rho]$,

for any $t, u \in X^+/\rho$, $T, U \in \mathsf{P}_f(X^+/\rho)$, $f \in \mathsf{End}(X^+/\rho)$, $q, q_1, \dots, q_k \in X^+$, $x, y \in X$.

Let $\sim \in \mathsf{Fic}X^\square$. We define a binary relation $\rho_\sim$ on X^+ in the following way: for any $q, r \in X^+$,

$$q\rho_\sim r \iff \{q\} \sim \{r\}.$$

Obviously, $\rho_\sim$ is a congruence on X^+. Further, we define an operator $[\ ]_\sim$: $\mathsf{P}(X^+/\rho_\sim) \to \mathsf{P}(X^+/\rho_\sim)$ in the following way: for any $T \in \mathsf{P}(X^+/\rho_\sim), q \in X^+$,

$$q\rho_\sim \in [T]_\sim \iff \begin{array}{l}\{q_1, \dots, q_k, q\} \sim \{q_1, \dots, q_k\} \\ \text{for some natural number } k \text{ and for } q_1, \dots, q_k \in X^+ \\ \text{such that } q_1\rho_\sim, \dots, q_k\rho_\sim \in T.\end{array}$$

Note that the definition of the operator $[\ ]_\sim$ is correct. Let $q, r, q_1, \dots, q_k \in X^+, q\rho_\sim r, \{q_1, \dots, q_k, q\} \sim \{q_1, \dots, q_k\}, q_1\rho_\sim, \dots, q_k\rho_\sim \in T$. Since $\{q\} \sim \{r\}$, we get $\{q_1, \dots, q_k, q\} \sim \{q_1, \dots, q_k, r\}$ and so $\{q_1, \dots, q_k, r\} \sim \{q_1, \dots, q_k\}$.

Now we can formulate the main theorem of [7].

Result 2.2. ([7, Theorem 4.7])

(i) *Let $\sim \in \mathsf{Fic}X^\square$. Then $\rho_\sim \in \mathsf{Fic}X^+$ and $[\ ]_\sim$ is a $\rho_\sim$-admissible closure operator.*
(ii) *Let $\rho \in \mathsf{Fic}X^+$ and let $[\ ]$ be a ρ-admissible closure operator. Let us define a binary relation $\sim_{\rho,[\ ]}$ on $\mathsf{P}_f(X^+)$ in the following way: for any $Q, R \in \mathsf{P}_f(X^+)$ we put*

$$Q \sim_{\rho,[\ ]} R \iff [Q\rho] = [R\rho].$$

Then $\sim_{\rho,[\ ]} \in \mathsf{Fic}X^\square$.
(iii) *For any $\sim \in \mathsf{Fic}X^\square$ it is the case that $\sim = \sim_{\rho_\sim,[\ ]_\sim}$.*
(iv) *For any $\rho \in \mathsf{Fic}X^+$ and any ρ-admissible closure operator $[\ ]$ it is the case that*

$$\rho = \rho_{\sim_{\rho,[\ ]}}, [\ ] = [\ ]_{\sim_{\rho,[\ ]}}.$$

Remark 2.3. Let $\mathcal{V}$ be a variety of semigroups. Then $\boldsymbol{SLO}\mathcal{V}$ is the variety determined by the set of identities $\{(\{u\},\{v\}) \mid (u,v) \in \rho_{\mathcal{V}}\}$. Let $\mathcal{W}$ be a variety of semilattice-ordered semigroups, $\sim$ be the corresponding fully invariant congruence on $X^{\square}$. By Result 2.2, $\sim$ corresponds to an ordered pair $(\rho, [\ ])$, where $\rho \in \mathsf{Fic}X^+$ and $[\ ]$ is a ρ-admissible closure operator (in the notation of Result 2.2, $\sim\, = \,\sim_{\rho,[\ ]}$, $\rho = \rho_\sim$, $[\ ] = [\ ]_\sim$). Now,

$$\begin{aligned} \mathcal{W} \subseteq \boldsymbol{SLO}\mathcal{V} &\iff \sim\, \supseteq \{(\{u\},\{v\}) \mid (u,v) \in \rho_{\mathcal{V}}\} \\ &\iff \rho \supseteq \rho_{\mathcal{V}}. \end{aligned}$$

So, finding all subvarieties in $\boldsymbol{SLO}\mathcal{V}$ is reduced to the description of all ρ-admissible closure operators for all $\rho \in \mathsf{Fic}X^+$ such that $\rho \supseteq \rho_{\mathcal{V}}$.

The following result gives the construction of the intersection of fully invariant congruences on $X^{\square}$ in terms of pairs $(\rho, [\ ])$.

Result 2.4. ([7, Theorem 4.9]) *Let $\rho_1, \rho_2 \in \mathsf{Fic}X^+$, $[\ ]_1$ be a ρ_1-admissible closure operator, $[\ ]_2$ be a ρ_2-admissible closure operator. Put $\sim\, = \,\sim_{\rho_1,[\ ]_1} \cap \sim_{\rho_2,[\ ]_2}$. Then*

$$\rho_\sim \;=\; \rho_1 \cap \rho_2,$$

$$q\rho_\sim \in [\{q_1\rho_\sim, \ldots, q_k\rho_\sim\}]_\sim \iff \begin{aligned} q\rho_1 &\in [\{q_1\rho_1, \ldots, q_k\rho_1\}]_1, \\ q\rho_2 &\in [\{q_1\rho_2, \ldots, q_k\rho_2\}]_2 \end{aligned}$$

$(q, q_1, \ldots, q_k \in X^+)$.

The operator $[\ ]_\sim$ from Result 2.4 is denoted by $[\ ]_1 \wedge [\ ]_2$. In this notation,

$$\sim_{\rho_1,[\ ]_1} \cap \sim_{\rho_2,[\ ]_2} = \sim_{\rho_1\cap\rho_2,[\ ]_1\wedge[\ ]_2}.$$

3. All admissible partial orders for varieties of left normal bands

Let $\rho \in \mathsf{Fic}X^+$. A binary relation $\leq$ on X^+/ρ is called a *ρ-admissible partial order* (or only an admissible partial order if ρ is known from the context) if it is a partial order on X^+/ρ which is compatible with the semigroup operation on X^+/ρ and satisfies the following conditions:

(A) $u \leq v$ implies $f(u) \leq f(v)$,

(B) $q\rho \leq r\rho,\ x \notin \mathsf{c}(r),\ s \in q(x|y)$ implies $s\rho \leq r\rho$

for any $u, v \in X^+/\rho$, $f \in \mathsf{End}(X^+/\rho)$, $q, r, s \in X^+$, $x, y \in X$.

The reason for introducing this notion follows from the next result. It will help us by finding ρ-admissible closure operators.

Result 3.1. ([7, Theorem 5.1]) *Let $\rho \in \mathsf{Fic}X^+$, $[\,]$ be a ρ-admissible closure operator. Let us define on X^+/ρ a binary relation $\leq$ in this way: for any $u, v \in X^+/\rho$,*

$$u \leq v \iff u \in [\{v\}].$$

Then the relation $\leq$ is a ρ-admissible partial order.

We recall basic facts concerning varieties of normal bands. At first, we present defining identities:

$$\boldsymbol{T} : x = y,\ \boldsymbol{LZ} : xy = x,\ \boldsymbol{RZ} : xy = y,\ \boldsymbol{SL} : x^2 = x, xy = yx,$$
$$\boldsymbol{LNB} : xyz = xzy,\ \boldsymbol{RNB} : xyz = yxz,\ \boldsymbol{ReB} : xyx = x,\ \boldsymbol{NB} : xyzx = xzyx.$$

We also use several operators on words from X^+. Let $u \in X^+$. Then

- $\mathsf{c}(u)$ is the set of all variables in u,
- $\mathsf{h}(u)$ is the first variable of u,
- $\mathsf{t}(u)$ is the last variable of u.

Result 3.2. ([5, p. 124])) *The lattice of all varieties of normal bands consists of 8 varieties mentioned above. The order by the inclusion is given by the diagram in Figure 1.*

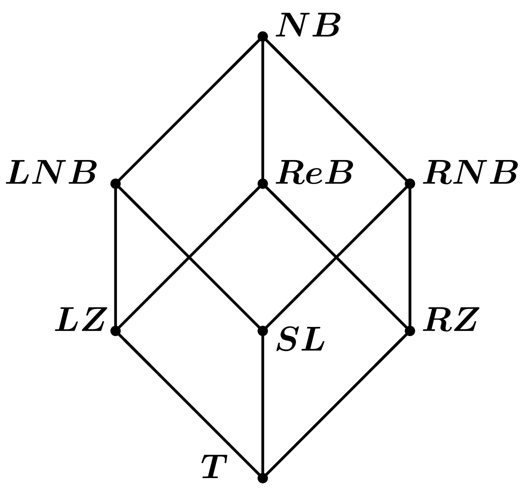

Figure 1. The lattice $\mathcal{L}(\boldsymbol{NB})$

Remark 3.3. Let $u, v \in X^+$. Then

(i) $u\rho_{\boldsymbol{LZ}}v$ if and only if $\mathsf{h}(u) = \mathsf{h}(v)$, $X^+/\rho_{\boldsymbol{LZ}} \cong (X, \circ)$ where $x \circ y = x$,

(ii) $u\rho_{\boldsymbol{SL}}v$ if and only if $\mathsf{c}(u) = \mathsf{c}(v)$, $X^+/\rho_{\boldsymbol{SL}} \cong (\mathsf{P}_f(X), \cup)$,

(iii) $u\rho_{\boldsymbol{LNB}}v$ if and only if $\mathsf{c}(u) = \mathsf{c}(v), \mathsf{h}(u) = \mathsf{h}(v)$, $X^+/\rho_{\boldsymbol{LNB}} \cong (\{(y, Y) | Y \in \mathsf{P}_f(X), y \in Y\}, \circ)$ where $(y, Y) \circ (z, Z) = (y, Y \cup Z)$, and the isomorphism is given by $u\rho_{\boldsymbol{LNB}} \mapsto (\mathsf{h}(u), \mathsf{c}(u))$.

The presented isomorphisms will be often used in such a way that isomorphic structures will be identified.

Theorem 3.4. ([6, Theorem 4.2]) *The relation* $=$ *is the only* $\rho_{\boldsymbol{LZ}}$*-admissible partial order.*

Theorem 3.5. ([6, Theorem 4.3], [7, Section 6]) *There are exactly three* $\rho_{\boldsymbol{SL}}$*-admissible partial orders, namely* $=, \subseteq, \supseteq$*.*

Now, we define three binary relations on $X^+/\rho_{\boldsymbol{LNB}}$. Let $(y, Y), (z, Z) \in X^+/\rho_{\boldsymbol{LNB}}$. We put

$$(y, Y) \leq_1 (z, Z) \Longleftrightarrow (y, Y) = (z, Z),$$
$$(y, Y) \leq_2 (z, Z) \Longleftrightarrow y = z, Y \subseteq Z,$$
$$(y, Y) \leq_3 (z, Z) \Longleftrightarrow y = z, Y \supseteq Z.$$

Theorem 3.6. *There exist exactly three* $\rho_{\boldsymbol{LNB}}$*-admissible partial orders, namely* $\leq_1, \leq_2, \leq_3$*.*

Proof. (a) Obviously, $\leq_1, \leq_2, \leq_3$ are three distinct binary relations on $X^+/\rho_{\boldsymbol{LNB}}$. It is clear that $\leq_1, \leq_2, \leq_3$ are partial orders which are compatible with the operation on $X^+/\rho_{\boldsymbol{LNB}}$ and satisfy (A).

Since the order $\leq_1$ is the identity, it is rather obvious that it satisfies the condition (B).

We will show that $\leq_2$ satisfies (B):

Let $q, r, s \in X^+, x, y \in X, x \notin \mathsf{c}(r), s \in q(x|y), \mathsf{h}(q) = \mathsf{h}(r), \mathsf{c}(q) \subseteq \mathsf{c}(r)$. We want to show that $\mathsf{h}(s) = \mathsf{h}(r)$, $\mathsf{c}(s) \subseteq \mathsf{c}(r)$. We see that $x \notin \mathsf{c}(q)$. So, $q(x|y) = \{q\}$, $s = q$, $\mathsf{h}(s) = \mathsf{h}(q)$, $\mathsf{c}(s) = \mathsf{c}(q)$. It follows that $\mathsf{h}(s) = \mathsf{h}(r)$, $\mathsf{c}(s) \subseteq \mathsf{c}(r)$.

We will show that $\leq_3$ satisfies (B):

Let $q, r, s \in X^+$, $x, y \in X$, $x \notin \mathsf{c}(r)$, $s \in q(x|y)$, $\mathsf{h}(q) = \mathsf{h}(r)$, $\mathsf{c}(q) \supseteq \mathsf{c}(r)$. We want to show that $\mathsf{h}(s) = \mathsf{h}(r)$, $\mathsf{c}(s) \supseteq \mathsf{c}(r)$. Since $\mathsf{h}(q) = \mathsf{h}(r)$, $x \notin \mathsf{c}(r)$, we have $\mathsf{h}(q) \neq x$. Then $\mathsf{h}(s) = \mathsf{h}(q)$, $\mathsf{h}(s) = \mathsf{h}(r)$. Further, $\mathsf{c}(s) = \mathsf{c}(q)$ or $\mathsf{c}(s) = (\mathsf{c}(q) - \{x\}) \cup \{y\}$ or $\mathsf{c}(s) = \mathsf{c}(q) \cup \{y\}$. Thus $\mathsf{c}(s) \supseteq \mathsf{c}(q) - \{x\}$. The fact $x \notin \mathsf{c}(r)$ implies $\mathsf{c}(q) - \{x\} \supseteq \mathsf{c}(r)$. So, $\mathsf{c}(s) \supseteq \mathsf{c}(r)$.

(b) Auxiliary statement 1: *Let* $\leq$ *be a* $\rho_{\boldsymbol{LNB}}$*-admissible partial order. Then* $(y, Y) \leq (z, Z)$ *implies* $y = z$*.*

Proof. Let $(y, Y) \leq (z, Z)$. Then $(y, Y)(z, Y \cup Z) \leq (z, Z)(z, Y \cup Z)$, $(y, Y \cup Z) \leq (z, Y \cup Z)$. Suppose that $y \neq z$. Let us apply an endomorphism $f \in \mathsf{End}(X^+/\rho_{\boldsymbol{LNB}})$ such that $f((y, \{y\})) = (z, \{z\})$, $f((z, \{z\})) = (y, \{y\})$, $f((u, \{u\})) = (u, \{u\})$ for $u \in (Y \cup Z) - \{y, z\}$. Then, by (A), $(z, Y \cup Z) \leq (y, Y \cup Z)$. It follows that $(y, Y \cup Z) = (z, Y \cup Z)$, $y = z$. It is a contradiction.

Auxiliary statement 2: *Let* $\leq$ *be a* $\rho_{\boldsymbol{LNB}}$*-admissible partial order. Then* $(y, Y) \leq (z, Z)$ *implies* $Y \subseteq Z$ *or* $Z \subseteq Y$*.*

Proof. Let $(y,Y) \le (z,Z)$. Suppose that $Y \not\subseteq Z, Z \not\subseteq Y$, i.e. $Y - Z \neq \emptyset, Z - Y \neq \emptyset$. By Auxiliary statement 1, $y = z \in Y \cap Z$. Let $p, q \in X, p \neq q$. Let us apply on $(y,Y) \le (z,Z)$ an endomorphism $f \in \mathsf{End}(X^+/\rho_{\boldsymbol{LNB}})$ such that $f((u,\{u\})) = (p,\{p\})$ for $u \in Y$, $f((v,\{v\})) = (q,\{q\})$ for $v \in Z - Y$. Then, by (A), $(p,\{p\}) \le (p,\{p,q\})$. Now, let us apply on $(y,Y) \le (z,Z)$ an endomorphism $g \in \mathsf{End}(X^+/\rho_{\boldsymbol{LNB}})$ such that $g((u,\{u\})) = (q,\{q\})$ for $u \in Y - Z, g((v,\{v\})) = (p,\{p\})$ for $v \in Z$. Then, by (A), $(p,\{p,q\}) \le (p,\{p\})$. So, $(p,\{p\}) = (p,\{p,q\}), \{p\} = \{p,q\}$. It is a contradiction. Auxiliary statement 2 is proved.

Now, let $\le$ be a $\rho_{\boldsymbol{LNB}}$-admissible partial order, $(a,A) \le (b,B), (a,A) \neq (b,B)$. By Auxiliary statement 1, $a = b$. By Auxiliary statement 2, exactly one of the following holds:
Case 1: $A \subset B$. We will show that $\le_2$ is $\le$. Let $(y,Y) \le_2 (z,Z)$. It means $y = z, Y \subseteq Z$. Let us apply on $(a,A) \le (b,B)$ an endomorphism $f \in \mathsf{End}(X^+/\rho_{\boldsymbol{LNB}})$ such that $f((u,\{u\})) = (y,Y)$ for $u \in A$, $f((v,\{v\})) = (z,Z)$ for $v \in B - A$. Then, by (A), $(y,Y) \le (y, Y \cup Z) = (z,Z)$. Let $(y,Y) \le (z,Z)$. By Auxiliary statement 1, $y = z$. We want to show that $Y \subseteq Z$. Suppose that $Z \subseteq Y$. So, $(z,Z) \le_2 (y,Y)$. We have just proved that it implies $(z,Z) \le (y,Y)$ But then $(y,Y) = (z,Z), Y = Z, Y \subseteq Z$. Now, let $Z \not\subseteq Y$. By Auxiliary statement 2, $Y \subseteq Z$.
Case 2: $B \subset A$. We have that $\le_3$ is $\le$. The idea of the proof of Case 2 is the same as that of the Case 1. So we omit this part of the proof. ∎

4. All admissible closure operators for varieties of left normal bands

Lemma 4.1. *Let $\rho \in \mathsf{Fic}X^+, \rho \supseteq \rho_{\boldsymbol{B}}, [\ ]$ be a ρ-admissible closure operator. Then, for any $u, v \in X^+/\rho$, $uv \in [\{u,v\}]$.*

Proof. Let $x, y \in X$, $x \neq y$. We have $x^2\rho \in [\{x\rho\}]$, $x^2(x|y) = \{x^2, xy, yx, y^2\}$, $x(x|y) = \{x, y\}$. By (IV), $\{x^2, xy, yx, y^2\}\rho \subseteq [\{x,y\}\rho]$. Now, $u = a\rho, v = b\rho$ for some $a, b \in X^+$. The fact $(x\rho)(y\rho) \in [\{x\rho, y\rho\}]$ implies (by (III), for $f \in \mathsf{End}(X^+/\rho)$, $f(x\rho) = a\rho$, $f(y\rho) = b\rho$) that $f(x\rho)f(y\rho) \in [\{f(x\rho), f(y\rho)\}]$, $(a\rho)(b\rho) \in [\{a\rho, b\rho\}]$, $uv \in [\{u,v\}]$. ∎

Theorem 4.2. *There exists exactly one $\rho_{\boldsymbol{T}}$-admissible closure operator, namely the identity operator on $\mathsf{P}(X^+/\rho_{\boldsymbol{T}})$.*

Proof. There is exactly one closure operator on a one-element set, which preserves the empty set, namely the identity operator. ∎

Theorem 4.3. ([6, Theorem 5.2]) *There exists exactly one $\rho_{\boldsymbol{LZ}}$-admissible closure operator, namely the identity operator on* $\mathsf{P}(X^+/\rho_{\boldsymbol{LZ}})$.

Now, we define four operators on $\mathsf{P}(X^+/\rho_{\boldsymbol{SL}})$. Let $T \in \mathsf{P}(X^+/\rho_{\boldsymbol{SL}}), Y \in X^+/\rho_{\boldsymbol{SL}}$. We put

$$\begin{aligned}
Y \in [T]_1 &\iff Z \subseteq Y \text{ for some } Z \in T,\\
Y \in [T]_2 &\iff Y \subseteq \textstyle\bigcup_{Z \in T} Z,\\
Y \in [T]_3 &\iff Z \subseteq Y \subseteq \textstyle\bigcup_{Z \in T} Z \text{ for some } Z \in T,\\
Y \in [T]_4 &\iff Y = Z_1 \cup \dots \cup Z_k \text{ for some } Z_1, \dots, Z_k \in T.
\end{aligned}$$

Theorem 4.4. ([6, Theorem 5.3], [7, Section 6]) *There exist exactly four $\rho_{\boldsymbol{SL}}$-admissible closure operators, namely* $[\ ]_1, [\ ]_2, [\ ]_3, [\ ]_4$.

Lemma 4.5. *Let $[\ ]$ be a $\rho_{\boldsymbol{LNB}}$-admissible closure operator. Let $(y, Y) \in X^+/\rho_{\boldsymbol{LNB}}$, $T \in \mathsf{P}(X^+/\rho_{\boldsymbol{LNB}}), (y, Y) \in [T]$. Then $y = z$ for some $(z, Z) \in T$.*

Proof. Since $[\ ]$ is a finitary closure operator, there are $(z_1, Z_1), \dots, (z_k, Z_k) \in T$ with $(y, Y) \in [\{(z_1, Z_1), \dots, (z_k, Z_k)\}]$. According to (II), $(y, Y)(z_1, Z_1 \cup \dots \cup Z_k) = (y, Y \cup Z_1 \cup \dots \cup Z_k) \in [\{(z_1, Z_1), \dots, (z_k, Z_k)\}][\{(z_1, Z_1 \cup \dots \cup Z_k)\}] \subseteq [\{(z_1, Z_1 \cup \dots \cup Z_k), (z_2, Z_1 \cup \dots \cup Z_k), \dots, (z_k, Z_1 \cup \dots \cup Z_k)\}]$. Let us assume that $y \notin \{z_1, \dots, z_k\}$. Let us consider the endomorphism $f \in \mathsf{End}(X^+/\rho_{\boldsymbol{LNB}})$ such that $f((y, \{y\})) = (y, \{y\})$, $f((u, \{u\})) = (z_1, \{z_1\})$ for $u \in X - \{y\}$. It follows from (III) that $(y, Y') \in [\{(z_1, Z')\}]$ for some $Y', Z' \in \mathsf{P}_f(X)$. In accordance with Result 3.1 and Theorem 3.6, $(y, Y') \leq_i (z_1, Z')$ for some $i \in \{1, 2, 3\}$, $y = z_1$. It is a contradiction. Thus $y \in \{z_1, \dots, z_k\}$. ∎

Now, we define a few operators on $\mathsf{P}(X^+/\rho_{\boldsymbol{LNB}})$. Let $T \in \mathsf{P}(X^+/\rho_{\boldsymbol{LNB}})$, $(y, Y) \in X^+/\rho_{\boldsymbol{LNB}}$. We put

$$\begin{aligned}
(y, Y) \in [T]_5 &\iff y = z_1, Z_2 \subseteq Y \text{ for some } (z_1, Z_1), (z_2, Z_2) \in T,\\
(y, Y) \in [T]_6 &\iff y = z, Z \subseteq Y \text{ for some } (z, Z) \in T,\\
(y, Y) \in [T]_7 &\iff y = z, Y \subseteq \textstyle\bigcup_{(z,Z) \in T} Z \text{ for some } (z, Z) \in T,\\
(y, Y) \in [T]_8 &\iff y = z_1, Z_2 \subseteq Y \subseteq \textstyle\bigcup_{(z,Z) \in T} Z \text{ for some,}\\
&\qquad (z_1, Z_1), (z_2, Z_2) \in T,\\
(y, Y) \in [T]_9 &\iff y = z, Z \subseteq Y \subseteq \textstyle\bigcup_{(z,Z) \in T} Z \text{ for some } (z, Z) \in T,\\
(y, Y) \in [T]_{10} &\iff y = z, Y = Z_1 \cup \dots \cup Z_k \text{ for some,}\\
&\qquad (z, Z), (z_1, Z_1), \dots, (z_k, Z_k) \in T,\\
(y, Y) \in [T]_{11} &\iff y = z_1, Y = Z_1 \cup \dots \cup Z_k \text{ for some,}\\
&\qquad (z_1, Z_1), \dots, (z_k, Z_k) \in T.
\end{aligned}$$

Theorem 4.6. *There exist exactly seven $\rho_{\boldsymbol{LNB}}$-admissible closure operators, namely* $[\ ]_5, [\ ]_6, [\ ]_7, [\ ]_8, [\ ]_9, [\ ]_{10}, [\ ]_{11}$.

Proof. (a) At first, we will prove that $[\ ]_6$ is a $\rho_{\boldsymbol{LNB}}$-admissible closure operator. It is almost obvious that $[\ ]_6$ is a finitary closure operator on $X^+/\rho_{\boldsymbol{LNB}}$ which preserves the empty set.

(I): Let $[\{(y,Y)\}]_6 = [\{(z,Z)\}]_6$. Then $(y,Y) \in [\{(y,Y)\}]_6 = [\{(z,Z)\}]_6$. It follows that $y = z, Z \subseteq Y$. Similarly, $z = y, Y \subseteq Z$. We have proved that $y = z, Y = Z$.

(II): Let $T, U \in \mathsf{P}_f(X^+/\rho_{\boldsymbol{LNB}})$. We want to show that $[T]_6[U]_6 \subseteq [TU]_6$. Let $(v,V) \in [T]_6$, $(w,W) \in [U]_6$. We have to prove that $(v,V)(w,W) = (v, V \cup W) \in [TU]_6$. There are $(z_1, Z_1) \in T$, $(z_2, Z_2) \in U$, $v = z_1$, $Z_1 \subseteq V$, $w = z_2$, $Z_2 \subseteq W$. It is $(z_1,Z_1)(z_2,Z_2) = (z_1, Z_1 \cup Z_2) \in TU$, $v = z_1$, $Z_1 \cup Z_2 \subseteq V \cup W$. It follows that $(v, V \cup W) \in [TU]_6$.

(III): Let $T \in \mathsf{P}_f(X^+/\rho_{\boldsymbol{LNB}})$, $f \in \mathsf{End}(X^+/\rho_{\boldsymbol{LNB}})$. We have to prove that $f([T]_6) \subseteq [f(T)]_6$. Let $(u,U) \in f([T]_6)$. We want to show that $(u,U) \in [f(T)]_6$. There is $(y,Y) \in [T]_6$, $(u,U) = f((y,Y))$. There is $(z,Z) \in T$, $y = z$, $Z \subseteq Y$. Let $(v,V) = f((z,Z))$. Then $(v,V) \in f(T)$. Since $(z,Z)(y,Y) = (z, Z \cup Y) = (y,Y)$, we have $f((y,Y)) = f((z,Z)(y,Y)) = f((z,Z))f((y,Y))$. Thus $(u,U) = (v,V)(u,U) = (v, V \cup U)$. Consequently, $u = v$, $U = V \cup U$, $V \subseteq U$. We have proved that $u = v$, $V \subseteq U$, $(v,V) \in f(T)$. Thus $(u,U) \in [f(T)]_6$.

(IV): Let $q, q_1, \dots, q_k \in X^+, x, y \in X$. Let $q\rho_{\boldsymbol{LNB}} \in [\{q_1\rho_{\boldsymbol{LNB}}, \dots, q_k\rho_{\boldsymbol{LNB}}\}]_6$. We want to show that $q(x|y)\rho_{\boldsymbol{LNB}} \subseteq [q_1(x|y)\rho_{\boldsymbol{LNB}} \cup \dots \cup q_k(x|y)\rho_{\boldsymbol{LNB}}]_6$. There is $i \in \{1, \dots, k\}, \mathsf{h}(q) = \mathsf{h}(q_i), \mathsf{c}(q) \supseteq \mathsf{c}(q_i)$. Let $r \in q(x|y)$. We have to prove that $r\rho_{\boldsymbol{LNB}} = (\mathsf{h}(r), \mathsf{c}(r)) \in [q_1(x|y)\rho_{\boldsymbol{LNB}} \cup \dots \cup q_k(x|y)\rho_{\boldsymbol{LNB}}]_6$.

We distinguish two cases: (A) $\mathsf{h}(q) \neq x$ (B) $\mathsf{h}(q) = x$.

Let r_i be a word which results from q_i by replacing all occurrences of the variable x with the variable y. It is $r_i \in q_i(x|y)$.

ad (A):

Case $y \notin \mathsf{c}(r)$: We have $r = q$ and clearly $(\mathsf{h}(r), \mathsf{c}(r)) = (\mathsf{h}(q), \mathsf{c}(q)) \in [q_1(x|y)\rho_{\boldsymbol{LNB}} \cup \dots \cup q_k(x|y)\rho_{\boldsymbol{LNB}}]_6$.

Case $y \in \mathsf{c}(r)$: We have $\mathsf{h}(r) = \mathsf{h}(q) = \mathsf{h}(q_i) = \mathsf{h}(r_i)$, $\mathsf{c}(r) \supseteq \mathsf{c}(q) - \{x\} \supseteq \mathsf{c}(q_i) - \{x\}$, $\mathsf{c}(r) \cup \{y\} \supseteq (\mathsf{c}(q_i) - \{x\}) \cup \{y\} \supseteq \mathsf{c}(r_i)$, $\mathsf{c}(r) \supseteq \mathsf{c}(r_i)$.

We see that $(\mathsf{h}(r), \mathsf{c}(r)) \in [q_1(x|y)\rho_{\boldsymbol{LNB}} \cup \dots \cup q_k(x|y)\rho_{\boldsymbol{LNB}}]_6$.

ad (B):

Case $\mathsf{h}(r) = x$: We know that $\mathsf{h}(r) = \mathsf{h}(q) = \mathsf{h}(q_i), \mathsf{c}(r) \supseteq \mathsf{c}(q) \supseteq \mathsf{c}(q_i)$. It implies $(\mathsf{h}(r), \mathsf{c}(r)) \in [q_1(x|y)\rho_{\boldsymbol{LNB}} \cup \dots \cup q_k(x|y)\rho_{\boldsymbol{LNB}}]_6$.

Case $\mathsf{h}(r) = y$: We know that $\mathsf{h}(r) = \mathsf{h}(r_i)$, $\mathsf{c}(r) \supseteq (\mathsf{c}(q) - \{x\}) \cup \{y\} \supseteq (\mathsf{c}(q_i) - \{x\}) \cup \{y\} = \mathsf{c}(r_i)$.

Thus $(\mathsf{h}(r), \mathsf{c}(r)) \in [q_1(x|y)\rho_{\boldsymbol{LNB}} \cup \dots \cup q_k(x|y)\rho_{\boldsymbol{LNB}}]_6$.

We have proved that $[\ ]_6$ is a $\rho_{\boldsymbol{LNB}}$-admissible closure operator. Note that

$$[\ ]_5 = \mathrm{id}_{\mathsf{P}(X^+/\rho_{\boldsymbol{LZ}})} \wedge [\ ]_1, \qquad [\ ]_7 = \mathrm{id}_{\mathsf{P}(X^+/\rho_{\boldsymbol{LZ}})} \wedge [\ ]_2, \qquad [\ ]_8 = [\ ]_7 \wedge [\ ]_5,$$
$$[\ ]_9 = [\ ]_6 \wedge [\ ]_8, \qquad [\ ]_{10} = [\ ]_8 \wedge [\ ]_4, \qquad [\ ]_{11} = [\ ]_9 \wedge [\ ]_{10}.$$

Till now, we have shown that $[\ ]_5, [\ ]_6, [\ ]_7, [\ ]_8, [\ ]_9, [\ ]_{10}, [\ ]_{11}$ really are $\rho_{\boldsymbol{LNB}}$-admissible closure operators.

Now we show that all operators $[\ ]_5 - [\ ]_{11}$ are different. At first we determine corresponding admissible partial orders to these operators. Let us notice that for $i = 5, 6$, $(y, Y) \in [\{(z, Z)\}]_i \iff (y, Y) \le_3 (z, Z)$, $(y, Y) \in [\{(z, Z)\}]_7 \iff (y, Y) \le_2 (z, Z)$, and for $i = 8, 9, 10, 11$, $(y, Y) \in [\{(z, Z)\}]_i \iff (y, Y) \le_1 (z, Z)$.

Now, let $x, y, z \in X$ be three different elements. Then we can see that

$$(y, \{x, y\}) \in [\{(x, \{x\}), (y, \{y, z\})\}]_5 - [\{(x, \{x\}), (y, \{y, z\})\}]_6,$$
$$(x, \{x, y\}) \in [\{(x, \{x, y, z\}), (y, \{x, y\})\}]_8 - [\{(x, \{x, y, z\}), (y, \{x, y\})\}]_9,$$
$$(x, \{x, y\}) \in [\{(x, \{x\}), (y, \{y, z\})\}]_8 \cap [\{(x, \{x\}), (y, \{y, z\})\}]_9,$$
$$(x, \{x, y\}) \notin [\{(x, \{x\}), (y, \{y, z\})\}]_{10} \cup [\{(x, \{x\}), (y, \{y, z\})\}]_{11},$$
$$(x, \{x, y\}) \in [\{(x, \{x, y, z\}), (y, \{x, y\})\}]_{10} - [\{(x, \{x, y, z\}), (y, \{x, y\})\}]_{11}.$$

(b) Finally, we show that each $\rho_{\boldsymbol{LNB}}$-admissible closure operator is equal to one of $[\ ]_5 - [\ ]_{11}$. By Theorem 3.6 and Result 3.1 we have that exactly one of the following cases occurs:

Case 1: $[\{(z, Z)\}] = \{(y, Y) \in X^+/\rho_{\boldsymbol{LNB}} \mid y = z, Y \subseteq Z\}$ for any $(z, Z) \in X^+/\rho_{\boldsymbol{LNB}}$. It is not difficult to show that $[\ ] = [\ ]_7$.

Case 2: $[\{(z, Z)\}] = \{(y, Y) \in X^+/\rho_{\boldsymbol{LNB}} \mid y = z, Y \supseteq Z\}$ for any $(z, Z) \in X^+/\rho_{\boldsymbol{LNB}}$. Let $T \in \mathsf{P}(X^+/\rho_{\boldsymbol{LNB}})$.

$[T]_6 \subseteq [T]$: Let $(y, Y) \in X^+/\rho_{\boldsymbol{LNB}}, (y, Y) \in [T]_6$. There is $(z, Z) \in T, y = z, Y \supseteq Z$. Then $(y, Y) \in [\{(z, Z)\}] \subseteq [T]$.

$[T] \subseteq [T]_5$: Let $(y, Y) \in X^+/\rho_{\boldsymbol{LNB}}, (y, Y) \in [T]$. By Lemma 4.5, there exists $(z, Z) \in T, y = z$. Further, there exist $(z_1, Z_1), \ldots, (z_k, Z_k) \in T, (y, Y) \in [\{(z_1, Z_1), \ldots, (z_k, Z_k)\}]$. Let us assume that $Z_1 \not\subseteq Y, Z_2 \not\subseteq Y, \ldots, Z_k \not\subseteq Y$. By (II), $(y, Y)(y, Y) = (y, Y) \in [\{(y, Y)\}][\{(z_1, Z_1), \ldots, (z_k, Z_k)\}] \subseteq [\{(y, Y \cup Z_1), \ldots, (y, Y \cup Z_k)\}]$. Let $x \in X, x \neq y$. Let us consider the endomorphism $f \in \mathsf{End}(X^+/\rho_{\boldsymbol{LNB}})$ such that $f((u, \{u\})) = (y, \{y\})$ for $u \in Y$, $f((v, \{v\})) = (x, \{x\})$ for $v \in X - Y$. Using (III) we obtain $(y, \{y\}) \in [\{(y, \{x, y\})\}]$. But then $\{y\} \supseteq \{x, y\}$, which is a contradiction.

Let $[\ ] \neq [\ ]_6$. So, there are $(z_1, Z_1), \ldots, (z_k, Z_k) \in X^+/\rho_{\boldsymbol{LNB}}$, $[\{(z_1, Z_1), \ldots, (z_k, Z_k)\}]_6 \subset [\{(z_1, Z_1), \ldots, (z_k, Z_k)\}]$. We will show that $[\ ] = [\ ]_5$. So, we have to prove that $[T]_5 \subseteq [T]$ for any $T \in \mathsf{P}(X^+/\rho_{\boldsymbol{LNB}})$. Let $(y, Y) \in [T]_5$.

There exist $(p,P),(q,Q)\in T, y=p, Y\supseteq Q$.
If $Y\supseteq P$ then $(y,Y)\in[\{(p,P)\}]\subseteq[T]$.
If $y=q$ then $(y,Y)\in[\{(q,Q)\}]\subseteq[T]$.
Now, let $Y\not\supseteq P$, $y\neq q$.
Let $(z,Z)\in[\{(z_1,Z_1),\dots,(z_k,Z_k)\}]-[\{(z_1,Z_1),\dots,(z_k,Z_k)\}]_6$. Note that if for some $1\le i\le k$, $z=z_i$ and $Z\supseteq Z_i$, then $(z,Z)\in[\{(z_1,Z_1),\dots,(z_k,Z_k)\}]_6$, which is contrary to the assumption. Therefore, there is no $1\le i\le k$ such that $z=z_i$ and $Z\supseteq Z_i$. We already know that $[\{(z_1,Z_1),\dots,(z_k,Z_k)\}]\subseteq[\{(z_1,Z_1),\dots,(z_k,Z_k)\}]_5$. It follows (without loss of generality) that $z=z_1$, $Z\supseteq Z_2$, $\neg(Z\supseteq Z_1)$, $z\neq z_2$, $z_1\neq z_2$. Let us consider the following endomorphism $f\in\mathsf{End}(X^+/\rho_{\boldsymbol{LNB}})$:

$$\begin{aligned}
&f((z,\{z\}))=f((z_1,\{z_1\}))=(y,\{y\})=(p,\{p\}),\\
&f((u,\{u\}))=(q,Y) &&\text{for } u\in Z-\{z\},\\
&f((v,\{v\}))=(p,P\cup Y) &&\text{for } v\in X-Z.
\end{aligned}$$

Based on (III) we obtain $f((z,Z))\in[\{f((z_1,Z_1)),\dots,f((z_k,Z_k))\}]$, $(y,Y)\in[\{(p,P\cup Y),(q,Y),(q,P\cup Y)\}]$. Recall that $(p,P),(q,Q)\in T$.
Since $P\cup Y\supseteq P$, we have $(p,P\cup Y)\in[\{(p,P)\}]\subseteq[T]$.
Since $Y\supseteq Q$, we have $(q,Y)\in[\{(q,Q)\}]\subseteq[T]$.
Since $P\cup Y\supseteq Y\supseteq Q$, we have $(q,P\cup Y)\in[\{(q,Q)\}]\subseteq[T]$.
Finally, $(y,Y)\in[[T]]\subseteq[T]$.
Case 3: $[\{(z,Z)\}]=\{(z,Z)\}$ for any $(z,Z)\in X^+/\rho_{\boldsymbol{LNB}}$. It can be proved that $[\,]=[\,]_i$, where $i\in\{8,9,10,11\}$. We omit the proof since it is technical again and similar to previous proofs. ∎

5. Varieties of semilattice-ordered left normal bands

In fact, the lattice of varieties of semilattice-ordered left normal bands is already determined in Section 4 where all $\rho_{\boldsymbol{T}}$-, $\rho_{\boldsymbol{LZ}}$-, $\rho_{\boldsymbol{SL}}$-, $\rho_{\boldsymbol{LNB}}$-admissible closure operators are described.

Let $\mathcal{V}$ be a variety of semilattice-ordered semigroups determined by a pair $(\rho,[\,])$ and $\mathcal{V}'$ be a variety of semilattice-ordered semigroups determined by a pair $(\rho',[\,]')$. Then, by Result 2.4, $\mathcal{V}\vee\mathcal{V}'$ is determined by the pair $(\rho\cap\rho',[\,]\wedge[\,]')$.

Theorem 5.1. *The lattice of all varieties of semilattice-ordered left normal bands is distributive, consists of* 13 *elements and it is presented in Figure* 2.

Proof. See Figure 2. The distributivity of $\mathcal{L}(\boldsymbol{SLOLNB})$ is directly verifiable. ∎

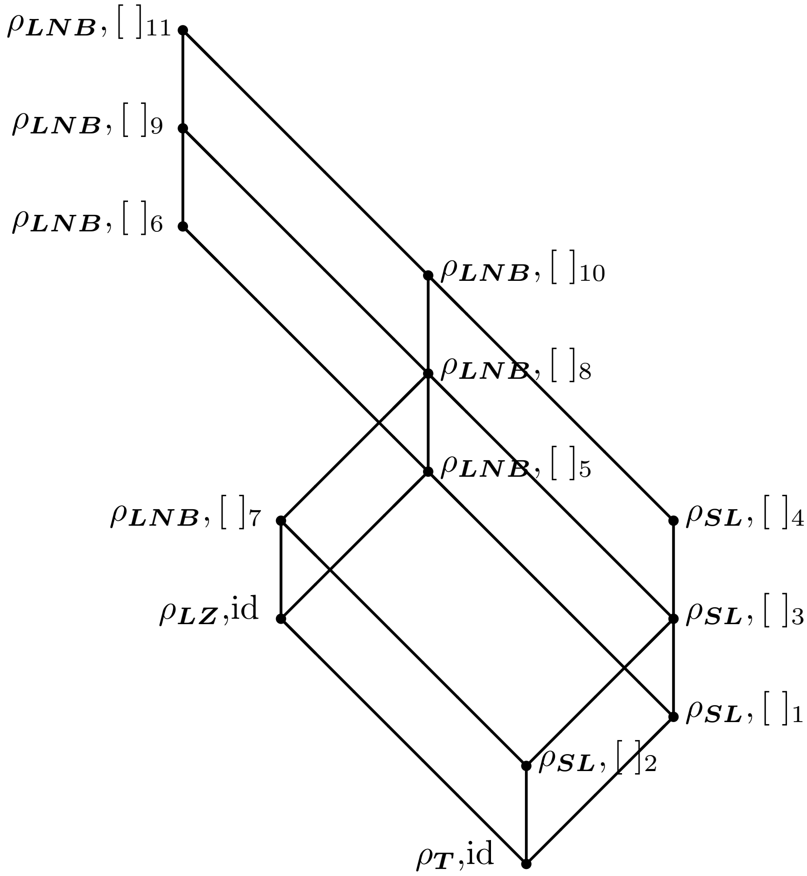

Figure 2. The lattice $\mathcal{L}(\boldsymbol{SLOLNB})$

$\mathcal{V}_1$	$(\rho_{\boldsymbol{T}}, \mathrm{id})$	$\boldsymbol{SLOT}$
$\mathcal{V}_2$	$(\rho_{\boldsymbol{LZ}}, \mathrm{id})$	$\boldsymbol{SLOLZ}$
$\mathcal{V}_3$	$(\rho_{\boldsymbol{SL}}, [\ ]_1)$	$\boldsymbol{D}$
$\mathcal{V}_4$	$(\rho_{\boldsymbol{SL}}, [\ ]_2)$	$\boldsymbol{M}$
$\mathcal{V}_5$	$(\rho_{\boldsymbol{SL}}, [\ ]_3)$	$x^2 = x, xy = yx, x \vee yz = x \vee xz \vee yx \vee yz$
$\mathcal{V}_6$	$(\rho_{\boldsymbol{SL}}, [\ ]_4)$	$\boldsymbol{SLOSL}$
$\mathcal{V}_7$	$(\rho_{\boldsymbol{LNB}}, [\ ]_5)$	$x^2 = x, xyz = xzy, x = x \vee xy, xy \vee yxz = yx \vee xyz$
$\mathcal{V}_8$	$(\rho_{\boldsymbol{LNB}}, [\ ]_6)$	$x^2 = x, xyz = xzy, x = x \vee xy$
$\mathcal{V}_9$	$(\rho_{\boldsymbol{LNB}}, [\ ]_7)$	$x^2 = x, xyz = xzy, x \vee xy \vee xyz = x \vee xyz,$ $xy \vee xyz \vee yxz = yx \vee xyz \vee yxz, xy = x \vee xy$
$\mathcal{V}_{10}$	$(\rho_{\boldsymbol{LNB}}, [\ ]_8)$	$x^2 = x, xyz = xzy, x \vee xy \vee xyz = x \vee xyz,$ $xy \vee xyz \vee yxz = yx \vee xyz \vee yxz$
$\mathcal{V}_{11}$	$(\rho_{\boldsymbol{LNB}}, [\ ]_9)$	$x^2 = x, xyz = xzy, x \vee xy \vee xyz = x \vee xyz$
$\mathcal{V}_{12}$	$(\rho_{\boldsymbol{LNB}}, [\ ]_{10})$	$x^2 = x, xyz = xzy, xy \vee xyz \vee yxz = yx \vee xyz \vee yxz$
$\mathcal{V}_{13}$	$(\rho_{\boldsymbol{LNB}}, [\ ]_{11})$	$\boldsymbol{SLOLNB}$

Table 1. Subvarieties of $\boldsymbol{SLOLNB}$ and their defining identities

Remark 5.2. It could be interesting to present varieties of semilattice-ordered left normal bands by admissible closure operators and simultaneously also by defining identities. We present this correspondence in Table 1. We denote by $\boldsymbol{D}$ the class of all distributive lattices and by $\boldsymbol{M}$ the class of all monobisemilattices i.e. semilattice-ordered semigroups satisfying the identities $x^2 = x, xy = yx, x \vee y = xy$. Elements from the variety $\boldsymbol{M}$ are sometimes called stammered semilattices. To obtain defining identities for varieties of semilattice-ordered left normal bands, we consulted articles [4] and [9].

6. The identity problem for all varieties of semilattice-ordered normal bands

We define an operator on $\mathsf{P}(X^+/\rho_{\boldsymbol{RNB}})$. Let $T \in \mathsf{P}(X^+/\rho_{\boldsymbol{RNB}})$, $(Y, y) \in X^+/\rho_{\boldsymbol{RNB}}$. We put

$$(Y, y) \in [T]_{6'} \iff y = z, Z \subseteq Y \text{ for some } (Z, z) \in T.$$

It follows from the dual version of Theorem 4.6 that $[\ ]_{6'}$ is a $\rho_{\boldsymbol{RNB}}$-admissible closure operator.

Denote by Irr the set of all join-irreducible varieties in $\mathcal{L}(\boldsymbol{SLONB})$.

Lemma 6.1. *The lattice of all varieties of semilattice-ordered normal bands contains 7 join-irreducible varieties. The partially ordered set $(Irr, \subseteq)$ is drawn in Figure 3.*

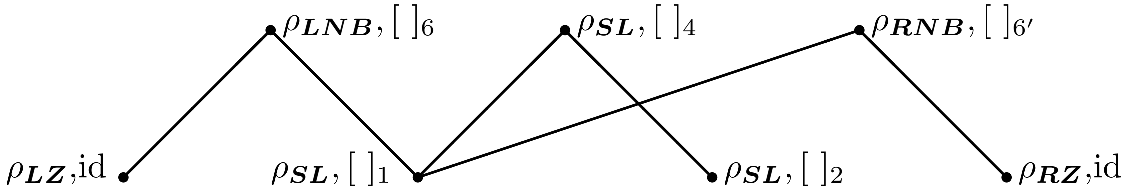

Figure 3. The partially ordered set $(Irr, \subseteq)$

Proof. Let $\mathcal{V}$ be a join-irreducible variety of semilattice-ordered normal bands. We know that $\boldsymbol{SLOLNB} \vee \boldsymbol{SLORNB} = \boldsymbol{SLONB}$ (see [9, Figure 3]) and that the lattice $\mathcal{L}(\boldsymbol{SLOB})$ is distributive (see [8]). Consequently,

$$\mathcal{V} = (\mathcal{V} \cap \boldsymbol{SLOLNB}) \vee (\mathcal{V} \cap \boldsymbol{SLORNB}).$$

Since $\mathcal{V}$ is join-irreducible, $\mathcal{V} = \mathcal{V} \cap \boldsymbol{SLOLNB}$ or $\mathcal{V} = \mathcal{V} \cap \boldsymbol{SLORNB}$, $\mathcal{V} \subseteq \boldsymbol{SLOLNB}$ or $\mathcal{V} \subseteq \boldsymbol{SLORNB}$. We use Theorem 5.1 describing the lattice $\mathcal{L}(\boldsymbol{SLOLNB})$ (see Figure 2) and the dual version of Theorem 5.1 describing the lattice $\mathcal{L}(\boldsymbol{SLORNB})$. ∎

Recall that a subset B of an ordered set $(A, \leq)$ is hereditary if $b \in B$, $a \in A$, $a \leq b$ implies $a \in B$.

Remark 6.2. It is shown in [8] that the lattice $\mathcal{L}(\boldsymbol{SLOB})$ is distributive. It follows that the lattice $\mathcal{L}(\boldsymbol{SLONB})$ of all varieties of semilattice-ordered normal bands is distributive. The partially ordered set $(Irr, \subseteq)$ of all join-irreducible varieties of the lattice $\mathcal{L}(\boldsymbol{SLONB})$, which uniquely determines the lattice $\mathcal{L}(\boldsymbol{SLONB})$, is drawn in Figure 3. The varieties of semilattice-ordered normal bands correspond to the 35 hereditary subsets of $(Irr, \subseteq)$.

Remark 6.3. Recall that a solution of the identity problem for a given variety consists in an effective description of identities valid in that variety. Let $\mathcal{V}$ be a variety of semilattice-ordered normal bands. Denote by $\{(\sigma_1, \langle\ \rangle_1), (\sigma_2, \langle\ \rangle_2), \ldots, (\sigma_n, \langle\ \rangle_n)\}$ the corresponding hereditary subset of $(Irr, \subseteq)$ and by $\sim$ the corresponding fully invariant congruence on $X^{\Box}$. Thus

$$\sim\, = \,\sim_{\sigma_1, \langle\ \rangle_1} \cap \sim_{\sigma_2, \langle\ \rangle_2} \cap \cdots \cap \sim_{\sigma_n, \langle\ \rangle_n} .$$

Let $Q, R \in \mathsf{P}_f(X^+)$. We have

$$\begin{aligned} Q \sim R \quad &\Longleftrightarrow \quad Q \sim_{\sigma_i, \langle\ \rangle_i} R \text{ for } i = 1, 2, \ldots, n \\ &\Longleftrightarrow \quad \langle Q\sigma_i \rangle_i = \langle R\sigma_i \rangle_i \text{ for } i = 1, 2, \ldots, n. \end{aligned}$$

By Lemma 6.1, $(\sigma_i, \langle\ \rangle_i)$ belong to the set

$$\{(\rho_{\boldsymbol{LZ}}, \mathrm{id}), (\rho_{\boldsymbol{SL}}, [\ ]_1), (\rho_{\boldsymbol{SL}}, [\ ]_2), (\rho_{\boldsymbol{RZ}}, \mathrm{id}), (\rho_{\boldsymbol{LNB}}, [\ ]_6), (\rho_{\boldsymbol{SL}}, [\ ]_4), (\rho_{\boldsymbol{RNB}}, [\ ]_{6'})\}$$

for $i = 1, 2, \ldots, n$. We have $Q\sigma_i, R\sigma_i \in \mathsf{P}_f(X^+/\sigma_i)$ and we decide in finitely many steps whether $\langle Q\sigma_i \rangle_i = \langle R\sigma_i \rangle_i$ or not.

So, we can solve the identity problem for any variety $\mathcal{V}$ of semilattice-ordered normal bands.

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M. Kuřil, Jan Evangelista Purkyne University, Faculty of Science, Department of Mathematics, Ceske mladeze 8, 400 96 Usti nad Labem, Czech Republic; *e-mail*: martin.kuril@ujep.cz

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