Introduction

The publication "Systems of equations and methods of their solving" is primarily intended for students of teacher training of mathematics at the Faculty of Science, Palacký University in Olomouc as a supporting text for a special seminar "Methods of solving mathematical problems" and also for those interested in solving non–standard mathematical problems. It focuses on elementary methods of solving more complex systems of nonlinear algebraic equations - i.e. methods that do not use the means of higher mathematics (mainly numerical methods of algebra). In Czech literature, we can encounter similar issues to a greater extent, especially in the publications [3] or [9] and in the older book [16].
The aim of this text is to better explain and describe the topic to the future (but also beginning) secondary school teachers of mathematics, both in terms of professional and didactic aspects. Various approaches are stressed to solve individual tasks. The presented tasks go beyond the scope of ordinary school mathematics, because it is possible to comment on this issue in more details from different viewpoints in their solutions (applied methods of solving). In the text you will find in total 60 challenging tasks. The first 25 of them contains complete solutions. The tasks are taken from Czech and foreign sources, especially from mathematical competitions for secondary school pupils, including several original tasks prepared by the author. The publication also contains a relatively large number of solved and unsolved problems focused on so–called cyclic systems of algebraic equations, as well as on special systems of algebraic equations that contain fewer equations (conditions) than unknowns. The set of 35 unsolved problems, which are presented in the final part of the publication, will serve to the reader to practice the methods of solving independently. All unsolved tasks are nevertheless provided with results.

Finally, I would like to thank Ms. Lucie Vaňková for substantial help with translating the text into English.

The second extended edition of the text has been supported of the project "Ukazatel P – pedagogické studijní programy", 2022.

Olomouc, June 2022

Author

Elementary symbols used in the text

$\mathbb{N}$ — set of positive integers, $\mathbb{N} = \{1, 2, 3, \dots\}$

$\mathbb{N}_0$ — set of non–negative integers

$\mathbb{Z}$ — set of integers

$\mathbb{R}$ — set of real numbers

$a \in A$ — a is an element of A

$a \notin A$ — a is not an element of A

$\emptyset$ — empty set

$A \subset B$ — A is a subset of a set B

$A \cup B$ — union of sets A and B

$A \cap B$ — intersection of sets A and B

$P \wedge Q$ — (logical) conjunction of propositions P and Q

$P \vee Q$ — (logical) disjunction of propositions P and Q

(x, y, z) — ordered triple x, y, z

$\max\{x, y\}$ — maximum of the numbers x and y

$\min\{x, y\}$ — minimum of the numbers x and y

$a \mid b$ — number a is a divisor of b

$\langle c, d\rangle$ — closed interval from c to d

$(0; \pi)$ — open interval from 0 to π

MO — Czech and Slovak Mathematical Olympiad

IMO — International Mathematical Olympiad

MEMO — Middle European Mathematical Olympiad

CPS — Czech–Polish–Slovak competition

USAMO — USA Mathematical Olympiad

Contents

1 Math problems and their classification

Mathematical problems are usually used in mathematical lessons, but also in many mathematical applications beyond school practice. We classify them according to several aspects - mainly according to their *content*, *usage*, and their *difficulty*. One of the basic views when classifying mathematical problems is their division according to the *assigned problem*. From this viewpoint, we distinguish mathematical problems into two categories - *determination problems*, which are divided into:

- calculation ("Calculate ...", "Determine ...", etc.),
- construction (geometric problems) ("Construct ..."),
- problems that examine sets of all points that share a property ("Determine a locus satisfying ..."),

and *proofs*. Those are usually formulated "Prove that ..." . This group also includes so called *decision problems*, i.e. problems which need a hypothesis to be formulated first (decide whether the proposition is valid or not) and then prove it. These problems are introduced with words "Decide if/whether ...".

This text addresses solely solving methods of *specific calculation problems*, namely elementary solving methods of systems of equations.

To be complete let us remind you that we distinguish mathematical problems according to their *content* into algebraic, arithmetic and numerically theoretical, geometrical, combinatorial, statistical and mixed (e.g. problems from combinatorial geometry, analytical geometry etc.). Another criterion to classify mathematical problems is their usage, i.e. *their usage in lessons*. We talk about motivational (e.g. introducing complex numbers), illustrative (during a lecture, explanation), practice (homework), diagnostic (tests), control and test tasks (large units, written school leaving exam in mathematics etc.). The last criterion to distinguish mathematical problems is the *difficulty* of the problems. We differentiate problems into common - unchallenging (suitable for immediate practice of discussed topic), moderately difficult (diagnostic, test tasks) and challenging problems (above-standard problems for mathematical competitions, etc.).

2 Systems of equations (solving methods)

This chapter studies nonstandard systems of equations, i.e. systems of equations that usually cannot be solved by standard methods, such as e.g. a system of linear equations or some types of simpler problems, which can be found in different secondary-school collections.

As you know, we use two elementary approaches to solve a system of equations: methods applying *equivalent manipulations* (the set of solutions of the given problem is equal to the set of solutions of the problem we get after we apply needed manipulations), and methods applying *non–equivalent manipulations*, where the set of solutions of the given problem is included in the set of solutions of the problem we get after manipulations. We need to check our solution after applying non–equivalent manipulation, as it is part of the solution. Checking the solution we eliminate those solutions, which do not meet the given (original) problem. We will highlight this fact every time.

To solve these problems we often use the following special methods, primarily:

- addition method,
- multiplication method,
- elimination method,
- sum of squares method,
- factoring method (product-form solution),
- method of inequalities and estimating equations,
- substitution method,
- geometrical method (using graphs of functions),
- extremal principle (i.e. the largest, smallest element).

Individual special methods are used in solving the following problems. As we can see, to solve some of those problems we need to combine and use simultaneously several special methods. The authors are solving some of the problems in the text by several different methods.

The following chapter introduces complete solutions of twenty-three nonstandard systems of equations, which represent the topic. The last (4^{th}) chapter includes unsolved problems meant to practise the knowledge.

The above stated special methods were used for solving the problems as follows (numbers represent number of the solved problems):

- addition: 1 (2nd and 3rd way of solution), 2, 2A, 2B, 2C, 3, 4 (1st way of solution), 5, 8 (1st way of solution), 9, 12, 16, 18, 19, 24
- multiplication: 6, 23, 25
- elimination: 1 (1st and 2nd way of solution), 4 (1st and 2nd way of solution), 5
- sum of squares method: 1 (3rd way of solution), 2, 2A, 2B, 2C, 3, 9 (1st and 2nd way of solution), 12
- factoring: 1 (2nd way of solution), 4 (1st and 2nd way of solution), 5, 19, 22
- method of inequalities and estimating equations: 1 (4th way of solution), 2B, 6, 7, 8, 9 (3rd way of solution), 10, 11, 12, 13, 15 (2nd way of solution), 16, 17, 18, 20
- substitution (especially goniometric substitution): 14, 15, 21
- geometrical method (using graphs of functions): 1 (5th way of solution), 8 (3rd way of solution)
- extremal principle: 2B, 7, 8, 14, 17

To manage the solution of unsolved problems in chapter 4 and individual study we recommend studying individual special methods we use to solve the systems of equations in chapter 3.

3 Selected problems (complete solutions)

This chapter introduces complete solutions of twenty-three selected systems of equations. We offer more solutions for some systems. At the beginning we usually mention the special methods used to solve the system.

Problem 1

Solve the system of equations in real numbers

$$x^2 + 1 = 2y, \tag{1}$$

$$y^2 + 1 = 2x. \tag{2}$$

We will present five ways of solving the given system of two nonlinear equations, where we highlight the important differences applying those methods. We can expect that most inexperienced students will try to apply the method of elimination to solve the system.

Solution 1 (elimination)

We use equation (1) to express $y = \frac{1}{2}(x^2 + 1)$ and substitute y in equation (2). Simplifying the expression we get

$$\left(\frac{x^2 + 1}{2}\right)^2 + 1 = 2x,$$

$$x^4 + 2x^2 - 8x + 5 = 0.$$

So we gained a fourth degree algebraic equation, which cannot be solved using secondary–school knowledge. We experimentally find out that one of the roots is $x = 1$. Dividing the polynomial on the left side of the equation by the adequate factor $x - 1$ we get

$$(x^4 + 2x^2 - 8x + 5) : (x - 1) = x^3 + x^2 + 3x - 5.$$

The final polynomial corresponds with the following algebraic equation of the third degree

$$x^3 + x^2 + 3x - 5 = 0.$$

Also in this case we can experimentally find out that the root of this equation is $x = 1$. Now we will divide the polynomial by the factor $x - 1$ again and we get a quadratic equation

$$x^2 + 2x + 5 = 0$$

with the discriminant $D = -16$. Therefore the algebraic equation

$$x^4 + 2x^2 - 8x + 5 = 0$$

has a double root 1 and no other real roots. Considering the used equation $y = \frac{1}{2}(x^2 + 1)$ we know that $y = 1$.

CONCLUSION. The given system of equations has a single unique solution in real numbers, which is a pair $(x, y) = (1; 1)$. The method, we used to solve the system, did not apply any non–equivalent manipulations. Therefore we do not have to check the solution.

SOLUTION 2 (addition, elimination)

We multiply both sides of equation (2) by number -1 and then we add the resulting equation to the given equation (1). We get

$$x^2 - y^2 = 2y - 2x,$$

which results after easy manipulation in an equation in factored form

$$(x - y)(x + y + 2) = 0.$$

Let us solve the system of equations

$$x^2 + 1 = 2y, \tag{3}$$
$$(x - y)(x + y + 2) = 0, \tag{4}$$

which is equivalent to the given system of equations (1), (2).

Considering the factored form of the equation (4) with the right side equal to 0 we have to consider two possibilities:

(i) Let $x - y = 0$, i.e. $x = y$. If we substitute y in the equation (3), we get $x^2 + 1 = 2x$, i.e. $(x - 1)^2 = 0$. This equation has the double real root $x = 1$, and therefore its unique real solution is the pair $(x, y) = (1; 1)$.

(ii) Let $x + y + 2 = 0$, i.e. $y = -x - 2$. Then $x^2 + 1 = -2(x + 2)$ and after modifying the equation we get the following quadratic equation

$$x^2 + 2x + 5 = (x + 1)^2 + 4 = 0,$$

which cannot be solved in real numbers.

CONCLUSION. The unique solution of the given system of equations is therefore the pair of real numbers $(x, y) = (1; 1)$. In this case we only used equivalent manipulations, and therefore we do not have to check the solution.

Solution 3 (addition, sum of squares)

Adding up the equations (1) and (2), applying simple manipulations we get

$$\begin{aligned}(x^2+1)+(y^2+1)&=2x+2y,\\(x^2-2x+1)+(y^2-2y+1)&=0,\\(x-1)^2+(y-1)^2&=0.\end{aligned}$$

The last equation implies $x=1$ and $y=1$.

This solution utilizes *non–equivalent* manipulation (the last equation is not equivalent to the given system of equations – it is its consequence) and inclusion $\mathcal{K}\subset\mathcal{K}'$ applies for the set of solutions $\mathcal{K}$ of the given system of equations and for the set of solutions $\mathcal{K}'$ of the last equation. Therefore we need to check the solution in this case (is a compulsory part of the solution). It helps us prove that the pair $(x,y)=(1;1)$ is really a unique solution of the given system of equations, which corresponds with the previous result.

Solution 4 (method of inequalities and estimating equations)

We know that x and y are positive real numbers as all expressions on left sides of the given system of equations are positive (larger than or equal to $\frac{1}{2}$). Now we can apply the known inequality of arithmetic and geometric mean (i.e. AG inequality) for any pair of non–negative real numbers. It stands

$$2x=y^2+1\geq 2\sqrt{y^2\cdot 1}=2|y|=2y=x^2+1\geq 2\sqrt{x^2\cdot 1}=2|x|=2x,$$

i.e.

$$2x\geq 2y\geq 2x,$$

which implies $x=y$. Now we continue the same way as e.g. in the Solution 2. As we did not use any non–equivalent manipulation in this solution, we do not need to check it.

Solution 5 (graphic method, approach of analytical geometry)

The first equation of the system can be rewritten $y=\frac{1}{2}(x^2+1)$, which is an analytical expression of the parabola. Its graph is symmetrical with respect to the y-axis in the Cartesian coordinate system Oxy. Also equation (2) is $x=\frac{1}{2}(y^2+1)$ in analytical expression of the parabola, whose graph is symmetrical with respect to the x-axis. Moreover, both parabolas are symmetrical with respect to the axis of the quadrant I and III of the coordinate system, thus to the line $x=y$. Possible intersection (intersections) of these curves must lie *only* on this line. This analysis shows that both parabolas touch at the point $P\,[1;1]$. We get the same result as in the previous solutions. Point P then corresponds with the pair of real numbers (x,y), whose terms are identical with the coordinates of the point P.

Problem 2

Solve the system of equations in real numbers

$$x^2 - 3y + 4 = z, \tag{1}$$
$$y^2 - 3z + 4 = x, \tag{2}$$
$$z^2 - 3x + 4 = y. \tag{3}$$

Solution. Let us consider the sum of all three equations (1)–(3), we get

$$(x^2 - 4x + 4) + (y^2 - 4y + 4) + (z^2 - 4z + 4) = 0\,,$$

i.e.

$$(x-2)^2 + (y-2)^2 + (z-2)^2 = 0\,. \tag{4}$$

There exists a unique triple of real numbers satisfying the last equation, i.e. $(x, y, z) = (2; 2; 2)$.

Conclusion. With respect to the non–equivalent manipulation used, inclusion $\mathcal{K} \subset \mathcal{K}'$ applies for the set of solutions of the given problem $\mathcal{K}$ and for the set of solutions of the last equation $\mathcal{K}'$. Therefore we need to check the solution, which shows us that $\mathcal{K} = \mathcal{K}'$. The problem has a unique solution in real numbers, namely $(x, y, z) = (2; 2; 2)$.

Let us try to apply the method used (*sum of squares*) to some numerically modified variants of the preceding problem to show its advantages and disadvantages.

Problem 2A

Solve the system of equations in real numbers

$$x^2 - 3y + 3 = z, \tag{5}$$
$$y^2 - 3z + 4 = x, \tag{6}$$
$$z^2 - 3x + 5 = y. \tag{7}$$

Solution. If we add the equations (5)–(7), we get the equation (4), and therefore $\mathcal{K}' = \{(2; 2; 2)\}$. But if we check the solution (also needed part of the solution), we can easily see that the triple $(x, y, z) = (2; 2; 2)$ does not satisfy the given system.

Conclusion. The system of equations (5)–(7) *does not have* a solution in real numbers.

Problem 2B

Solve the system of equations in real numbers

$$x^2 - 3y + 3 = z, \tag{8}$$
$$y^2 - 3z + 3 = x, \tag{9}$$
$$z^2 - 3x + 3 = y. \tag{10}$$

If we add all three equations of this system (8)–(10), we get after manipulation

$$(x^2 - 4x + 4) + (y^2 - 4y + 4) + (z^2 - 4z + 4) = 3,$$

and thus

$$(x - 2)^2 + (y - 2)^2 + (z - 2)^2 = (\sqrt{3})^2.$$

The last equation is (from the viewpoint of analytical geometry) the equation of the spherical surface with the centre in the point $S\,[2; 2; 2]$ and the radius $r = \sqrt{3}$. The set $\mathcal{K}'$ is therefore non–finite (more precisely non–denumerable). As $\mathcal{K} \subset \mathcal{K}'$, we need to check the solution. However it is practically *impossible* and therefore we *cannot* precisely specify the set $\mathcal{K}$ of all solutions of the problem.

We can see that the sum of squares method has no universal application and its use is often limited by the given numerical parameters.

Note. If we examine the given cyclic system of equations in details, which is based on applying special methods of numerical mathematics for solving systems of nonlinear equations, we can surprisingly find out that the given system of equations has two real solutions, i.e. $(x, y, z) = (1; 1; 1)$ and $(x, y, z) = (3; 3; 3)$.[1]

DIFFERENT SOLUTION. As in the preceding solution we add the equations (8)–(10) and we get

$$(x - 2)^2 + (y - 2)^2 + (z - 2)^2 = 3. \tag{11}$$

Thus we can see that all components of the solution sought (x, y, z) are positive real numbers. If any of the components (e.g. x) was non–positive, then it applies for the corresponding summand on the left

$$(x - 2)^2 \geq 4 > 3,$$

which contradicts the equation (11).

To solve the system of equations (8)–(10) we can apply the combination of *the extremal principle* and *the method of inequalities and estimating equations.* We can assume that one of the components (e.g. x) is the largest (one of the largest). It is necessary to distinguish two cases:

---

[1] The given system of equations 2B can be solved in a different way – e.g. by *the method of inequalities and estimating equations*, see the solution below.

(i) Let $x \geq y \geq z > 0$. Given the assumption, then $x^2 \geq z^2$ and it applies

$$x^2 - 3y + 3 \geq z^2 - 3x + 3 = y \geq z = x^2 - 3y + 3.$$

Therefore, in all the inequalities used, equality must occur, i.e. $x = y = z$.

(ii) Let $x \geq z \geq y > 0$. As in (i) inequalities apply

$$x^2 - 3y + 3 \geq y^2 - 3z + 3 = x \geq z = x^2 - 3y + 3,$$

and therefore $x = y = z$.

In both cases we came to the same result $x = y = z$. Substituting this relation e.g. into the equation (8) of the given system, we get a quadratic equation

$$x^2 - 3x + 3 = x, \qquad \text{i.e.} \qquad x^2 - 4x + 3 = (x-1)(x-3) = 0$$

of the unknown x. The (real) roots of the last quadratic equation are $x_1 = 1$ and $x_2 = 3$.

Conclusion. The given system of equations (8)–(10) has exactly two solutions in $\mathbb{R}$, i.e. $(x_1, y_1, z_1) = (1; 1; 1)$ and $(x_2, y_2, z_2) = (3; 3; 3)$. We do not have to check the solution.

Problem 2C

Solve the system of equations in real numbers

$$x^2 - 3y + 5 = z, \tag{12}$$
$$y^2 - 3z + 5 = x, \tag{13}$$
$$z^2 - 3x + 5 = y. \tag{14}$$

Solution. If we add all three equations (12)–(14), we get after easy manipulation

$$(x^2 - 4x + 4) + (y^2 - 4y + 4) + (z^2 - 4z + 4) = -3\,,$$
$$(x-2)^2 + (y-2)^2 + (z-2)^2 = -3\,.$$

Conclusion. The last equation shows that the set of its solutions $\mathcal{K}' = \emptyset$. As $\mathcal{K} \subset \mathcal{K}'$, even the system of equations (12)–(14) does not have any real solution.

Note. The given problem can be further developed. We can e.g. show that the system of equations

$$x^2 - 3y + p = z,$$
$$y^2 - 3z + p = x,$$
$$z^2 - 3x + p = y,$$

with the real parameter p has exactly two real solutions for $1 \leq p \leq 4$, i.e.

$$x = y = z = 2 \pm \sqrt{4-p}.$$

Problem 3 ([23], J. Švrček)

Solve the system of equations in real numbers

$$\begin{aligned} 2x^2 + 2xy + 1 &= 4z, && (1)\\ 2y^2 + 2yz + 1 &= 4x, && (2)\\ 2z^2 + 2zx + 1 &= 4y. && (3) \end{aligned}$$

Solution. If we add all three equations (1)–(3), we get after easy manipulation

$$(x^2+y^2+1+2xy-2x-2y)+(y^2+z^2+1+2yz-2y-2z)+(z^2+x^2+1+2zx-2z-2x) =$$

$$= (x+y-1)^2 + (y+z-1)^2 + (z+x-1)^2 = 0. \qquad (4)$$

The equation (4) implies

$$x + y = y + z = z + x = 1.$$

Solving this system of three linear equations we find out that $x = y = z = \frac{1}{2}$. Since we applied the non–equivalent manipulation, we need to check the solution. We verify that the triple of real numbers found is also the solution of the given system of equations.

Conclusion. The given system of equations has a unique solution in $\mathbb{R}$, i.e. $(x, y, z) = (\frac{1}{2}; \frac{1}{2}; \frac{1}{2})$.

Problem 4

Solve the system of equations in real numbers

$$\begin{aligned} x^2 + y + z &= 2, && (1)\\ y^2 + z + x &= 2, && (2)\\ z^2 + x + y &= 2. && (3) \end{aligned}$$

Solution 1. If we multiply both sides of the equation (2) by -1 and add it to the equation (1), we get

$$x^2 - y^2 - (x - y) = 0$$

and we rewrite the left side of the equation into factored form

$$(x - y)(x + y - 1) = 0.$$

Similarly, the equations (1) and (3) suggest

$$x^2 - z^2 - (x - z) = 0,$$

i.e.

$$(x - z)(x + z - 1) = 0.$$

Thus the given system of equations (1)–(3) is equivalent to the system of equations

$$\begin{aligned} x^2 + y + z &= 2, && (4)\\ (x - y)(x + y - 1) &= 0, && (5)\\ (x - z)(x + z - 1) &= 0. && (6) \end{aligned}$$

Now we will proceed from the equations (5) and (6), in which we consider the following possibilities:

(i) $(y = x) \wedge (z = x)$. If we substitute y and z in the equation (4), we get $x^2+2x-2 = 0$, where $x_1 = -1+\sqrt{3}$ and $x_2 = -1-\sqrt{3}$. Solutions in this case are the triples of real numbers $(x_1, y_1, z_1) = (-1 + \sqrt{3}; -1 + \sqrt{3}; -1 + \sqrt{3})$ and $(x_2, y_2, z_2) = (-1 - \sqrt{3}; -1 - \sqrt{3}; -1 - \sqrt{3})$.

(ii) $(y = x) \wedge (z = 1 - x)$. If we substitute y and z in the equation (4), we get $x^2 - 1 = 0$, with the real roots $x_3 = -1$ and $x_4 = 1$. Solutions in this case are the triples $(x_3, y_3, z_3) = (-1; -1; 2)$ and $(x_4, y_4, z_4) = (1; 1; 0)$.

(iii) $(y = 1 - x) \wedge (z = x)$. If we substitute y and z in the equation (4), we get the quadratic equation $x^2 - 1 = 0$, with the real roots $x_5 = -1$ and $x_6 = 1$. Solutions in this case are the triples $(x_5, y_5, z_5) = (-1; 2; -1)$ and $(x_6, y_6, z_6) = (1; 0; 1)$.

(iv) $(y = 1 - x) \wedge (z = 1 - x)$. If we substitute y and z in the equation (4), we get $x^2 - 2x = 0$, with the real roots $x_7 = 2$ and $x_8 = 0$. Solutions in this case are the triples $(x_7, y_7, z_7) = (2; -1; -1)$ and $(x_8, y_8, z_8) = (0; 1; 1)$.

Conclusion. In total, we got eight real solutions of the given system, see (i)–(iv). As we used only equivalent manipulations while solving the problem, we do not need to check them.

Solution 2. We express $z = 2 - x^2 - y$ from the equation (1) and substitute z in the equations (2) and (3) by this value. We get the following system of two equations with two unknowns x and y

$$\begin{aligned} y^2 + (2 - x^2 - y) + x &= 2,\\ (2 - x^2 - y)^2 + x + y &= 2. \end{aligned}$$

After easy manipulation we get the system of equations

$$(y-x)(y+x-1)=0, \tag{7}$$
$$x^4+2x^2y-4x^2+x+y^2-3y+2=0. \tag{8}$$

With respect to the factored form of the equation (7) we have to consider two possibilities.

(i) Let $y=x$. If we substitute y in the equation (8), we get

$$x^4+2x^3-3x^2-2x+2=0,$$

i.e.

$$(x^2+2x-2)(x+1)(x-1)=0,$$

with the real roots $x_1=-1+\sqrt{3}$, $x_2=-1-\sqrt{3}$, $x_3=-1$ and $x_4=1$. All roots of the last fourth degree algebraic equation produce further components of the solutions $y_1=-1+\sqrt{3}$, $y_2=-1-\sqrt{3}$, $y_3=-1$ and $y_4=1$ and by using the equation $z=2-x^2-y$ we get $z_1=-1-\sqrt{3}$, $z_2=-1-\sqrt{3}$, $z_3=2$ and $z_4=0$.

(ii) Let $y=1-x$. If we substitute y in the equation (8), we get

$$x^4-2x^3-x^2+2x=0,$$

i.e.

$$(x+1)(x-1)(x-2)x=0,$$

with the real roots $x_5=-1$, $x_6=1$, $x_7=2$ and $x_8=0$. The roots of the last equation correspond with the solution components $y_5=2$, $y_6=0$, $y_7=-1$, $y_8=1$ and $z_5=-1$, $z_6=1$, $z_7=-1$, $z_8=1$.

This method led us to the same solution as in the first case. As we applied only equivalent manipulations we do not need to check the solution.

Problem 5 (addition, factorization)

Solve the system of equations in real numbers

$$x^2+y^2+z=2, \tag{1}$$
$$y^2+z^2+x=2, \tag{2}$$
$$z^2+x^2+y=2. \tag{3}$$

SOLUTION. If we subtract the equation (2) from the equation (1), we get

$$x^2-z^2-(x-z)=0$$

and after manipulation

$$(x-z)(x+z-1)=0.$$

If we subtract the equation (2) from the equation (3), we get

$$(x-y)(x+y-1)=0.$$

Let us consider the system of equations

$$x^2+y^2+z=2, \tag{4}$$

$$(x-y)(x+y-1)=0, \tag{5}$$

$$(x-z)(x+z-1)=0, \tag{6}$$

which is equivalent to the given system of equations (1)–(3). We will write factored forms of the left sides of the equations (5) and (6) zero right sides and we consider the next four possibilities:

(i) Let $(y=x)\wedge(z=x)$. If we substitute y and z in the equation (4) from both the equations, we get the following quadratic equation

$$2x^2+x-2=0,$$

its real roots are $x_1=\frac{1}{4}(-1+\sqrt{17})$ and $x_2=\frac{1}{4}(-1-\sqrt{17})$. These correspond with the following two solutions of the given system of equations

$$(x_1,y_1,z_1)=\left(\frac{-1+\sqrt{17}}{4};\frac{-1+\sqrt{17}}{4};\frac{-1+\sqrt{17}}{4}\right),$$

$$(x_2,y_2,z_2)=\left(\frac{-1-\sqrt{17}}{4};\frac{-1-\sqrt{17}}{4};\frac{-1-\sqrt{17}}{4}\right).$$

(ii) Let $(y=x)\wedge(z=1-x)$. If we substitute y and z in the equation (4) from both equations, we also get a quadratic equation

$$2x^2-x-1=0$$

with real roots $x_3=-\frac{1}{2}$ and $x_4=1$. These roots correspond with the following two solutions of the given system of equations

$$(x_3,y_3,z_3)=\left(-\frac{1}{2};-\frac{1}{2};\frac{3}{2}\right) \qquad \text{and} \qquad (x_4,y_4,z_4)=(1;1;0)\,.$$

(iii) Let $(y=1-x)\wedge(z=x)$. If we substitute y and z in the equation (4) from both equations, we get the same quadratic equation as in (ii)

$$2x^2-x-1=0.$$

The real roots $x_5=-\frac{1}{2}$, $x_6=1$ of this quadratic equation correspond with other two solutions of the given system of equations in this case, i.e.

$$(x_5,y_5,z_5)=\left(-\frac{1}{2};\frac{3}{2};-\frac{1}{2}\right) \qquad \text{and} \qquad (x_6,y_6,z_6)=(1;0;1)\,.$$

(iv) And finally, let $(y = 1 - x) \wedge (z = 1 - x)$. Now we can proceed to the solution of the quadratic equation

$$2x^2 - 3x = 0.$$

Its roots are real numbers $x_7 = \frac{3}{2}$ and $x_8 = 0$. These correspond with the last two solutions of the given system of equations

$$(x_7, y_7, z_7) = \left(\frac{3}{2}; -\frac{1}{2}; -\frac{1}{2}\right) \qquad \text{and} \qquad (x_8, y_8, z_8) = (0; 1; 1)\,.$$

CONCLUSION. The given system of equations has 8 solutions, which we have introduced in (i)–(iv). We do not have to check the solutions as we did not use any non–equivalent manipulation solving the problem.

Solving the following problem (Diophantine system of equations) we applied multiplicative method, whose final effect is similar to the one of factorization method.

Problem 6 (44th MO, 1994/1995, C–S–1, J. Švrček)
Solve the system of equations in non–negative integers

$$a + bc = 3c, \tag{1}$$
$$b + ca = 3a, \tag{2}$$
$$c + ab = 3b. \tag{3}$$

SOLUTION. Firstly, we notice that if one of the components of the solution (a, b, c) equals zero, then all components equal zero. So we found the solution $(a, b, c) = (0; 0; 0)$.

Let us suppose that $a, b, c > 0$. Further we can rewrite the given system of equations (1)–(3) in the form

$$a = c(3 - b), \tag{4}$$
$$b = a(3 - c), \tag{5}$$
$$c = b(3 - a). \tag{6}$$

If we multiply the equations (4)–(6), i.e. we use the *multiplication method*, in case $abc \neq 0$ we get the equation

$$(3 - a)(3 - b)(3 - c) = 1\,. \tag{7}$$

Each factor on the left side (7) can only be an integer, i.e. 1 or -1. If we analyse all four possibilities (i.e. we check the solution), we find another solution in positive integers, $(a, b, c) = (2; 2; 2)$. The other three possibilities do not have a solution of the given system of equations.

Conclusion. The given system of equations has two solutions in non–negative integers, $(a, b, c) = (0; 0; 0)$ and $(a, b, c) = (2; 2; 2)$.

Note. Other possible method of solving uses *divisibility* in non–negative integers. The equations (4)–(6) induce $(c \,|\, a) \wedge (a \,|\, b) \wedge (b \,|\, c)$, which leads us directly to $a = b = c$.

To solve the following two problems we will introduce so–called *extremal principle*, which can be broadly applied not only to solving systems of equations.

Problem 7

Solve the system of equations in non–negative real numbers

$$\begin{aligned} x_1 + x_2 &= x_3^2, \\ x_2 + x_3 &= x_4^2, \\ x_3 + x_4 &= x_5^2, \\ x_4 + x_5 &= x_1^2, \\ x_5 + x_1 &= x_2^2. \end{aligned}$$

Solution. Firstly, if any component of a solution of the system of equations equals zero, then the other four components equal zero as well. We found a trivial solution of the given system of equations $(x_1, x_2, x_3, x_4, x_5) = (0; 0; 0; 0; 0)$. Secondly, suppose $x_1 x_2 x_3 x_4 x_5 \neq 0$ and let us find all solutions of the given problem in positive real numbers. We will apply so–called *extremal principle.* Let $K = \max\{x_1, x_2, x_3, x_4, x_5\}$ and $L = \min\{x_1, x_2, x_3, x_4, x_5\}$. Then there exists $i \in \{1, 2, 3, 4, 5\}$ such that $x_i + x_{i+1} = K^2$, where $x_6 = x_1$. Therefore it holds

$$2K = K + K \geq x_i + x_{i+1} = K^2, \qquad \text{i.e.} \qquad 2K - K^2 = K(2 - K) \geq 0.$$

Since $K > 0$, simultaneously $K \leq 2$.

But at the same time it exists $j \in \{1, 2, 3, 4, 5\}$ such that $x_j + x_{j+1} = L^2$, where $x_6 = x_1$. Therefore it is necessarily true

$$2L = L + L \leq x_j + x_{j+1} = L^2, \qquad \text{i.e.} \qquad 2L - L^2 = L(2 - L) \leq 0.$$

Since $L > 0$, then $L \geq 2$. In total we get

$$2 \leq L = \min\{x_1, x_2, x_3, x_4, x_5\} \leq \max\{x_1, x_2, x_3, x_4, x_5\} = K \leq 2,$$

$$\text{i.e.} \quad 2 \leq L \leq K \leq 2.$$

It implies $K = L$ and therefore $(x_1, x_2, x_3, x_4, x_5) = (2; 2; 2; 2; 2)$.

Conclusion. We need to check the solution and make sure that the given system of equations has two solutions in non–negative real numbers, namely $(x_1, x_2, x_3, x_4, x_5) \in \{(0; 0; 0; 0; 0), (2; 2; 2; 2; 2)\}$.

Problem 8

Solve the system of equations in real numbers

$$
\begin{aligned}
(x+y)^3 &= z, & (1)\\
(y+z)^3 &= x, & (2)\\
(z+x)^3 &= y. & (3)
\end{aligned}
$$

Solution. We will apply *the extremal principle* (in this case *maximum*) in solving the system of equations again.

At first, we can easily find out that the system of equations has a solution $(x, y, z) = (0; 0; 0)$. If e.g. $x = 0$, then the equation (2) implies that $z = -y$ and if we use substitution in (3), we get $-y^3 = y$, i.e. $y + y^3 = y(y^2 + 1) = 0$, where $y = 0$, and therefore $z = 0$ as well.

Next let $xyz \neq 0$. Without loss of generality, assume that e.g. x is the largest component of the system of equations, i.e. $x \geq y$ and simultaneously $x \geq z$. If we assume $x \geq y$ and use the equations (2) and (3) of the given system of equations, it implies

$$(y+z)^3 \geq (z+x)^3, \quad \text{which implies} \quad y \geq x.$$

Therefore $x = y$.

Similarly, based on the assumption $x \geq z$ and using the equations (1) and (2) of the given system of equations we get

$$(y+z)^3 \geq (x+y)^3, \quad \text{thus} \quad z \geq x.$$

Therefore $z = x$.

In total we get $x = y = z$. Substituting y and z from the last relation in the equation (1) we get a cubic equation (algebraic equation of the third degree with real coefficients)

$$(2x)^3 = x \quad \text{and after manipulation we yield} \quad x(8x^2 - 1) = 0,$$

which has three real roots: 0, $\frac{1}{4}\sqrt{2}$ and $-\frac{1}{4}\sqrt{2}$. Considering the condition $xyz \neq 0$ we get two other non–zero solutions of the given system of equations.

Conclusion. The given system of equations has exactly three real solutions (x, y, z) and those are $(0; 0; 0)$, $(\frac{1}{4}\sqrt{2}; \frac{1}{4}\sqrt{2}; \frac{1}{4}\sqrt{2};)$ and $(-\frac{1}{4}\sqrt{2}; -\frac{1}{4}\sqrt{2}; -\frac{1}{4}\sqrt{2};)$. We do not have to check the solution.

The following three problems are unusual mainly because the number of equations is lower than the number of unknowns. Such a system is known as an *underdetermined system.* To solve such nonstandard problems we may take advantage of using special methods and techniques, see **Problems 9–11**.

Problem 9

Solve the system of equations in real numbers

$$x + y + z = 3, \tag{1}$$
$$x^2 + y^2 + z^2 = 3. \tag{2}$$

Solution 1. If we multiply both sides of the equation (1) by the number -2 and add it to the equation (2), we get

$$(x-1)^2 + (y-1)^2 + (z-1)^2 = 0,$$

which has a unique solution in real numbers $(x, y, z) = (1; 1; 1)$. As we applied a non–equivalent manipulation, we need to check the solution. We make sure that the triple $(1; 1; 1)$ is a solution of the given system of equations.

Conclusion. The given system of equations has exactly one solution in real numbers, which is the triple $(x, y, z) = (1; 1; 1)$.

Solution 2. We express $z = 3 - x - y$ from the equation (1) and we substitute z in the equation (2). We get

$$x^2 + y^2 + (3 - x - y)^2 = 3.$$

After manipulations we get

$$2x^2 + 2y^2 - 6x - 6y + 2xy + 6 = 0,$$

$$(x-1)^2 + (y-1)^2 + (x+y-2)^2 = 0.$$

Hence $x = 1$, $y = 1$ and with regard to the expression used $z = 3 - x - y$, then $z = 1$. The triple $(x, y, z) = (1; 1; 1)$ is a solution of the given problem. We do not need to check the solution as we did not apply non–equivalent manipulations.

Solution 3. The following solution will introduce a method applying means of analytical geometry. The equation (1) is an equation of a plane and the equation (2) is an equation of a sphere with its centre in the beginning of the Cartesian coordinate system Oxy and its radius $\sqrt{3}$. We only need to prove that the plane given by the equation (1) is a tangent plane to the sphere given by the equation (2) and its point of tangency is the point with the coordinates $[1; 1; 1]$.

SOLUTION 4. It applies the well–known Cauchy–Schwarz inequality, see e.g. [21] for the triples (x, y, z), $(1; 1; 1)$) of real numbers. It holds for the above-mentioned triples of real numbers according to the Cauchy–Schwarz inequality that

$$\underbrace{(x^2 + y^2 + z^2)}_{=3}\underbrace{(1^2 + 1^2 + 1^2)}_{=3} \geq \underbrace{(x \cdot 1 + y \cdot 1 + z \cdot 1)^2}_{=9}.$$

In this case the inequality converts to equality hence $x = y = z = \lambda$. Then we have $x+y+z = 3\lambda = 3$ hence $\lambda = 1$. Therefore the given system of equations has a unique real solution $(x, y, z) = (1; 1; 1)$. We do not need to check the solution.

We substantially apply so–called *method of inequalities and estimating equations* in solving the systems (6)–(9), as well as in solving the following three **Problems (10)–(12)**.

Problem 10

Solve the system of equations in real numbers

$$\begin{aligned} 2x^2 + (y + z)^2 &= 4 - t^2, & (1)\\ x^4 + 3(y + 1)^2 &= t - 2. & (2) \end{aligned}$$

SOLUTION. The left sides of both equations gain non–negative values, therefore we need to find all $t \in \mathbb{R}$ satisfying these conditions, i.e.

$$(4 - t^2 \geq 0) \wedge (t - 2 \geq 0),$$

and therefore

$$(-2 \leq t \leq 2) \wedge (t \geq 2).$$

Hence, it holds $t = 2$. If we substitute t, we get $x = 0$, $y = -1$ from the equation (2) and $z = 1$ from the equation (1). We do not need to check the solution considering the method we applied.

CONCLUSION. The unique solution of the given system of equations in real numbers is the quadruple $(x, y, z, t) = (0; -1; 1; 2)$.

Problem 11

Solve the system of equations in positive real numbers

$$\begin{aligned} x + y + z &= 6, & (1)\\ xyz &= 8. & (2) \end{aligned}$$

SOLUTION. If we apply the so–called AG-inequality (arithmetic–geometric means inequality) to a triple of non–negative real numbers x, y, z, we get

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz}.$$

Equality occurs if and only if $x = y = z$. Given by the task searching for a triple of positive real numbers x, y, z, it holds

$$2 = \frac{x+y+z}{3} \geq \sqrt[3]{xyz} = 2. \tag{3}$$

As equality holds for the inequality (3), it must hold $x = y = z = 2$ with respect to the equations (1) and (2).

CONCLUSION. The unique real solution of the given system of equations is the triple $(x, y, z) = (2; 2; 2)$. We do not need to check the solution considering the method we applied to solve the system.

Problem 12 (61$^{\text{st}}$ MO, 2011/2012, A–III–3, J. Švrček)

Solve the system of equations in real numbers

$$x^4 + y^2 + 4 = 5yz, \tag{1}$$
$$y^4 + z^2 + 4 = 5zx, \tag{2}$$
$$z^4 + x^2 + 4 = 5xy. \tag{3}$$

SOLUTION. Firstly, let us estimate the left side of the first equation of the given system of equations using the inequality $4x^2 \leq x^4 + 4$, which is true for any real number x. The equality holds if and only if $x^2 = 2$, i.e. if and only if $x = \sqrt{2}$ or $x = -\sqrt{2}$.

We get

$$4x^2 + y^2 \leq x^4 + y^2 + 4 = 5yz\,.$$

We also analogously estimate the left sides of the other two equations of this system. We get the following triple of inequalities

$$4x^2 + y^2 \leq 5yz, \tag{4}$$
$$4y^2 + z^2 \leq 5zx, \tag{5}$$
$$4z^2 + x^2 \leq 5xy. \tag{6}$$

If we add them up, we get the following inequality after simple manipulation

$$x^2 + y^2 + z^2 \leq xy + yz + zy\,, \tag{7}$$

and subsequently

$$(x-y)^2+(y-z)^2+(z-x)^2 \le 0\,. \tag{8}$$

Hence the equality necessarily occurs in both inequalities (7) and (8), i.e. it holds $x=y=z$. The equality must be true even for all inequalities (4)–(6). Hence

$$x=y=z=\sqrt{2} \quad \text{or} \quad x=y=z=-\sqrt{2}\,.$$

This time we need to check the solution and we can easily see that both found real solutions satisfy the given system of equations.

Conclusion. The given system of equations has exactly two solutions in real numbers and these are the triples $(x,y,z)\in\{(\sqrt{2};\sqrt{2};\sqrt{2}),(-\sqrt{2};-\sqrt{2};-\sqrt{2})\}$.

Problem 13 (7^{th} USAMO, 1978, [11])

Let a,b,c,d,e be real numbers satisfying the equations

$$\begin{aligned} a+b+c+d+e &= \ 8, & (1)\\ a^2+b^2+c^2+d^2+e^2 &= 16. & (2)\end{aligned}$$

Determine the maximum value of e.

Solution. If we use the Cauchy–Schwarz inequality for quadruples $(1;1;1;1)$, (a,b,c,d) of real numbers, we get

$$(1^2+1^2+1^2+1^2)(a^2+b^2+c^2+d^2) \ge (a+b+c+d)^2.$$

The preceding inequality and the conditions of the problem imply

$$4(16-e^2) \ge (8-e)^2, \qquad \text{i.e.} \qquad e(5e-16) \le 0\,.$$

All real numbers e, which satisfy $0 \le e \le \frac{16}{5}$, are the solutions of this inequality. Therefore the maximum value of e under the given conditions is $\frac{16}{5}$. Then equality occurs in the inequality, hence $a=b=c=d$. The equation (1) implies $a=b=c=d=\frac{6}{5}$.

Conclusion. The maximum value of e under the conditions is $\frac{16}{5}$. This value corresponds with the quintuple of real numbers $(a,b,c,d,e)=(\frac{6}{5};\frac{6}{5};\frac{6}{5};\frac{6}{5};\frac{16}{5})$, which satisfies the conditions of the problem.

To solve the following two problems we apply so–called *goniometric substitution method*, which is a dynamic way of solving difficult equations (inequalities) and their systems. (We will combine it with *the extremal principle* in **Problem 15**.)

Problem 14

Solve the following system of equations in real numbers

$$x_1 - \frac{1}{x_1} = 2x_2, \tag{1}$$
$$x_2 - \frac{1}{x_2} = 2x_3, \tag{2}$$
$$x_3 - \frac{1}{x_3} = 2x_4, \tag{3}$$
$$x_4 - \frac{1}{x_4} = 2x_1. \tag{4}$$

Solution. We apply the goniometric identity

$$2\,\operatorname{cotg} 2\alpha = \operatorname{cotg}\alpha - \frac{1}{\operatorname{cotg}\alpha}\,,$$

which holds for all $\alpha \neq \frac{k\pi}{2}$, where $k \in \mathbb{Z}$.

It is obvious that all x_i ($i = 1, 2, 3, 4$) are non–zero. Let us put $x_1 = \operatorname{cotg}\alpha$, i.e. we use so-called *goniometric substitution*, where $\alpha \in (0; \frac{1}{2}\pi) \cup (\frac{1}{2}\pi; \pi)$. The equation (1) implies $x_2 = \operatorname{cotg} 2\alpha$. The equation (2) analogously implies $x_3 = \operatorname{cotg} 4\alpha$, the third equation then $x_4 = \operatorname{cotg} 8\alpha$ and finally the last equation implies $x_1 = \operatorname{cotg} 16\alpha$. Hence it holds $x_1 = \operatorname{cotg}\alpha = \operatorname{cotg} 16\alpha$, i.e. $16\alpha - \alpha = k\pi$, where k is a suitable integer and therefore $\alpha = \frac{1}{15}k\pi$, where $k = 1, 2, \ldots, 14$.

Conclusion. All quadruples (x_1, x_2, x_3, x_4) of real numbers are the solutions of the given system, for which it holds

$$(x_1, x_2, x_3, x_4) = \left(\operatorname{cotg}\frac{k\pi}{15};\ \operatorname{cotg}\frac{2k\pi}{15};\ \operatorname{cotg}\frac{4k\pi}{15};\ \operatorname{cotg}\frac{8k\pi}{15}\right),$$

where $k = 1, 2, \ldots, 14$. We do not need to check the solution.

Note. It is obvious that all components of the solution of the given system are in pairs distinct real numbers and there is no solution with same components, which is rarely seen in cyclic systems of equations.

Problem 15 (55th MO, 2005/2006, A–III–6, J. Švrček, P. Calábek)

Solve the system of equations in real numbers

$$\operatorname{tg}^2 x + 2\operatorname{cotg}^2 2y = 1\,,$$
$$\operatorname{tg}^2 y + 2\operatorname{cotg}^2 2z = 1\,,$$
$$\operatorname{tg}^2 z + 2\operatorname{cotg}^2 2x = 1\,.$$

Solution. For every $\varphi \neq \frac{1}{2}k\pi$, where $k \in \mathbb{Z}$, it holds

$$2\ \operatorname{cotg}^2 2\varphi = 2\left(\frac{\cos^2\varphi - \sin^2\varphi}{2\sin\varphi\cos\varphi}\right)^2 = \frac{1}{2}\left(\operatorname{tg}^2\varphi + \operatorname{cotg}^2\varphi - 2\right).$$

Let us use $\operatorname{tg}^2 x = a$, $\operatorname{tg}^2 y = b$ and $\operatorname{tg}^2 z = c$, where a, b, c are positive real numbers. We rewrite the given system in the following form

$$a + \frac{1}{2}\left(b + \frac{1}{b}\right) = 2, \tag{1}$$

$$b + \frac{1}{2}\left(c + \frac{1}{c}\right) = 2, \tag{2}$$

$$c + \frac{1}{2}\left(a + \frac{1}{a}\right) = 2. \tag{3}$$

Without loss of generality, let us suppose that a is the largest number (one of the largest) in the set $\{a, b, c\}$. Let $a \geq b \geq c$ (we would use the same approach in case that $a \geq c \geq b$). Under the assumption the preceding system of equations implies

$$b + \frac{1}{b} \leq c + \frac{1}{c} \leq a + \frac{1}{a}. \tag{4}$$

As it holds $x + 1/x \geq 2$ for every positive real number x, the system of equations (1)–(3) implies

$$0 < a, b, c \leq 1.$$

Since the function $f(x) = x + 1/x$ is decreasing in the interval $(0; 1\rangle$, the inequality

$$a + \frac{1}{a} \leq b + \frac{1}{b} \leq c + \frac{1}{c}$$

holds. If we combine it with the inequality (4), we get $a = b = c$.

We need to determine all real numbers $u \in (0; 1\rangle$, which satisfy the equation

$$u + \frac{1}{2}\left(u + \frac{1}{u}\right) = 2.$$

After simple manipulation we get a quadratic equation

$$3u^2 - 4u + 1 = 0, \quad \text{i.e.} \quad (u - 1)(3u - 1) = 0.$$

This quadratic equation has two positive real roots $u_1 = 1$ and $u_2 = \frac{1}{3}$.

Conclusion. The solutions of the given system of equations are the triples (x, y, z) of real numbers with respect to the substitutions applied and the periodicity of the tangent function.

$$\left(\frac{\pi}{4} + k_1\frac{\pi}{2}, \frac{\pi}{4} + k_2\frac{\pi}{2}, \frac{\pi}{4} + k_3\frac{\pi}{2}\right) \quad \text{and} \quad \left(\pm\frac{\pi}{6} + \ell_1\pi, \pm\frac{\pi}{6} + \ell_2\pi, \pm\frac{\pi}{6} + \ell_3\pi\right),$$

where $k_1, k_2, k_3, \ell_1, \ell_2, \ell_3$ are any integers, and yet the three signs in the triple of the second solution can be selected arbitrarily (independently). We do not need to check the solution.

A specific position among the solved problems has the following system of equations. Both of its unknowns appear independently, as well as the arguments of goniometric functions. All the more surprising is the fact that this system can be solved applying elementary methods.

Problem 16 (55^{th} MO, 2005/2006, A–II–4, J. Švrček)

Solve the system of equations in real numbers

$$\begin{aligned} \sin^2 x + \cos^2 y &= y^2\,, \\ \sin^2 y + \cos^2 x &= x^2\,. \end{aligned}$$

Solution. First of all, we need to realize that with each solution (x, y) in real numbers, pairs $(x, -y)$, $(-x, y)$ and $(-x, -y)$ are also solutions of the system. Therefore we just have to restrict to non–negative real numbers. Moreover, if the solution of the system is a pair (x, y), a pair (y, x) is also the solution of the same system). We may further suppose that $0 \le x \le y$.

Next, manipulate both equations according to the relation $\cos^2 \alpha = 1 - \sin^2 \alpha$:

$$\begin{aligned} \sin^2 x + 1 - \sin^2 y &= y^2\,, \\ \sin^2 y + 1 - \sin^2 x &= x^2\,. \end{aligned}$$

If we add up both equations, we get

$$x^2 + y^2 = 2\,. \tag{1}$$

Subtracting the second equation from the first one, we have

$$2\sin^2 x - 2\sin^2 y = y^2 - x^2\,,$$

i.e.

$$2\,(\sin x + \sin y)\,(\sin x - \sin y) = y^2 - x^2\,. \tag{2}$$

Assuming $0 \le x \le y$, $0 \le x \le y \le \sqrt{2} < \frac{1}{2}\pi$ results from (1). As the function sine is non–negative and increasing on the interval $\langle 0; \frac{1}{2}\pi)$,we can see that for such real numbers x and y the left side of the equation (2) is non–positive and the right side is non–negative. It means that $y^2 - x^2 = 0$, which means $x = y$ under the assumption and $x = y = 1$ with respect to (1).

The given system of equations has a unique solution $(x, y) = (1; 1)$ in the non–negative real numbers. (We do not need to check the solution.)

CONCLUSION. The given system of equations has four solutions in real numbers. These are the pairs $(1;1)$, $(1;-1)$, $(-1;1)$ and $(-1;-1)$.

Problem 17 (CPS competition before 57th IMO, 2016)

Solve the system of equations in real numbers

$$\begin{aligned}(a+b)(a^2+b^2) &= (c+d)(c^2+d^2),\\ (a+c)(a^2+c^2) &= (b+d)(b^2+d^2),\\ (a+d)(a^2+d^2) &= (b+c)(b^2+c^2).\end{aligned}$$

SOLUTION. Let us denote $f(x,y)=(x+y)(x^2+y^2)$. We will prove that for any real numbers x, y, z, where $y \geq z$, the inequality $f(x,y) \geq f(x,z)$ holds. It holds

$$f(x,y)-f(x,z)=(x+y)(x^2+y^2)-(x+z)(x^2+z^2)=$$

$$=(x^2y-x^2z)+(xy^2-xz^2)+(y^3-z^3)=(y-z)(x^2+xy+xz+y^2+yz+z^2)=$$

$$=\frac{1}{2}(y-z)\left[(x+y)^2+(y+z)^2+(z+x)^2\right]\geq 0.$$

The equality occurs if $(y=z)\vee(x=y=z=0)$.

We can rewrite the given system of equations into a simplified form with respect to the introduced notation

$$\begin{aligned}f(a,b) &= f(c,d),\\ f(a,c) &= f(b,d),\\ f(a,d) &= f(b,c).\end{aligned}$$

The given system of equations is symmetric with respect to the unknowns a, b, c, d, therefore we can assume without loss of generality that $a=\max\{a,b,c,d\}$. It holds with respect to the symmetry of the function f in both variables

$$f(c,d)=f(a,b)\geq f(c,b)=f(a,d)\geq f(b,d)=f(a,c)\geq f(d,c)=f(c,d).$$

The equality occurs in all inequalities used if $a=b=c=d$.

CONCLUSION. The solution of the given system of equations is every quadruple (a,b,c,d) of real numbers in the form $(a,b,c,d)=(k,k,k,k)$, where k is any real number.

Note. We do not need to check the solution, because we used methods (*method of inequalities and estimating equations* and *extremal principle*) that do not require this. We can also apply analogous approach in solving the system of equations as in **Problem 4** and **5**.

Problem 18 (6^{th} MEMO, 2012, T–1, J. Švrček)

Solve the system of equations in real numbers

$$2x^3+1=3zx, \tag{1}$$
$$2y^3+1=3xy, \tag{2}$$
$$2z^3+1=3yz. \tag{3}$$

Solution. We will prove that the given system of equations has two real solutions, i.e. $(x,y,z)=(1;1;1)$ and $(x,y,z)=(-\frac{1}{2};-\frac{1}{2};-\frac{1}{2})$.

We will analyse the following four cases according to how many numbers (components of solution) x, y, z are non–negative.

(i) Let all components x, y, z be non–negative. If we add up all three equations of the given system, we get after simple manipulation

$$(x^3+y^3+1)+(y^3+z^3+1)+(z^3+x^3+1)=3xy+3yz+3zx. \tag{4}$$

If we apply AG inequality on the triple of non–negative real numbers x^3, y^3, 1, we get

$$x^3+y^3+1 \geq 3\sqrt[3]{x^3\cdot y^3\cdot 1}=3xy, \tag{5}$$

in this case the equality occurs if $x^3=y^3=1$, i.e. $x=y=1$. It analogously holds

$$y^3+z^3+1 \geq 3yz, \tag{6}$$
$$z^3+x^3+1 \geq 3zx. \tag{7}$$

The quality concurrently occurs in the inequalities (5)–(7) if $x=y=z=1$. Hence there is the only triple (x,y,z) of non–negative real numbers, which satisfies the equation (4), i.e $(x,y,z)=(1;1;1)$. We need to check the solution (we applied a non–equivalent manipulation) and we prove that the triple found satisfies also the given system of equations.

(ii) Exactly one of the numbers x, y, z is negative (the other two components are non–negative). As the given system of equations is cyclic, we can assume that $x<0$ and $y,z\geq 0$. Then it holds with respect to the equation (2)

$$1 \leq 2y^3+1=3xy\leq 0,$$

which is impossible. In this case the system of equations does not have a real solution.

(iii) Let $x,y<0$ and $z\geq 0$. Using the equation (3) we can easily prove as in the case (ii) that the system of equations has no real solution.

(iv) Let $x, y, z < 0$. Let us put $a = -x$, $b = -y$ and $c = -z$. If we substitute the positive numbers a, b, c in (1)–(3), they satisfy the cyclic system of equations

$$2a^3 + 3ca = 1, \tag{8}$$
$$2b^3 + 3ab = 1, \tag{9}$$
$$2c^3 + 3bc = 1. \tag{10}$$

We can assume without loss of generality that $a = \max\{a, b, c\}$. Then the inequalities $2a^3 \geq 2c^3$ and $3ca \geq 3bc$ hold. If we add them, we get with respect to (8) and (10)

$$2a^3 + 3ca \geq 2c^3 + 3bc = 2a^3 + 3ca.$$

We can see that equality must hold in both inequalities, i.e. $a = b = c$. If we substitute e.g. in (8), we get a cubic equation

$$2a^3 + 3a^2 - 1 = 0,$$

which can be rewritten in the form

$$2(a+1)^2\left(a - \frac{1}{2}\right) = 0.$$

This equation has a unique real root $a = \frac{1}{2}$. As we used substitution, we need to check the solution. We find out that (in this case) there is the only triple of negative numbers satisfying the given system of equations, i.e. $(x, y, z) = (-\frac{1}{2}; -\frac{1}{2}; -\frac{1}{2})$.

Conclusion. The given cyclic system of equations has exactly two real solutions, which we described in the parts (i) and (iv).

Problem 19 (Greek MO, 2014)

Solve the system of equations in real numbers

$$\begin{aligned} x^3 &= \frac{z}{y} - \frac{2y}{z}, \\ y^3 &= \frac{x}{z} - \frac{2z}{x}, \\ z^3 &= \frac{y}{x} - \frac{2x}{y}. \end{aligned}$$

Solution. We simplify the given system of equations to the equivalent form (for $xyz \neq 0$)

$$x^3yz = z^2 - 2y^2, \tag{1}$$
$$xy^3z = x^2 - 2z^2, \tag{2}$$
$$xyz^3 = y^2 - 2x^2. \tag{3}$$

If we add the equations (1)–(3), we get the only equation

$$xyz(x^2+y^2+z^2) = -(x^2+y^2+z^2)$$

and after simple manipulation

$$(x^2+y^2+z^2)(xyz+1) = 0.$$

As $xyz \neq 0$, it is $x^2+y^2+z^2 > 0$, and therefore it holds

$$xyz = -1. \tag{4}$$

If we use the equation (4) in (1)–(3), we get after manipulation

$$\begin{aligned} x^2 &= -z^2 + 2y^2, && (5)\\ y^2 &= -x^2 + 2z^2, && (6)\\ z^2 &= -y^2 + 2x^2. && (7) \end{aligned}$$

If we subtract the equations (6) from (5) and then (7) from (6), we find that $x^2 = y^2 = z^2$. When trying to get the solution of the given system of equations we need to consider four possible cases, either $x = y = \pm z$, or $x = -y = \pm z$.
If we install them in (4), we get four triples of real numbers $(-1;-1;-1)$, $(1;1;-1)$, $(1;-1;1)$ and $(-1;1;1)$. The check of solution (we need to use it as we applied non–equivalent manipulation) proves that all four triples satisfy the given system of equations.

Conclusion. The given system of equations has exactly four real solutions we introduced in the previous paragraph.

We may approach the solution of the following problem in different ways. All of them use manipulations of inequalities if needed to be combined with the extremal principle.

Problem 20

Solve the system of equations in real numbers

$$\begin{aligned} 3a &= (b+c+d)^3, && (1)\\ 3b &= (c+d+e)^3, && (2)\\ 3c &= (d+e+a)^3, && (3)\\ 3d &= (e+a+b)^3, && (4)\\ 3e &= (a+b+c)^3. && (5) \end{aligned}$$

Solution. Let us assume that $a < b$. Then it holds with respect to the equations (1) and (2) that $b < e$, i.e. $a < b < e$. Hence the equations (1) and (5) show that $d < a$ and in total $d < a < b < e$. Considering the equations (4) and (5) we get $e < c$. In total it holds

$$d < a < b < e < c. \tag{6}$$

Considering the equations (3) and (4) we get $b < d$, which is in contradiction with (6).
If $a > b$, we simultaneously get to a contradiction – in analogous relation to (6). Therefore the system of equations has not a real solution in the considered two cases.
The last possible case, where $a = b$, i.e.

$$a = b = c = d = e, \tag{7}$$

leads after substitution for b, c, d from (7) e.g. in (1) to the solution of the equation

$$3a = (3a)^3, \qquad \text{i.e.} \qquad a(9a^2 - 1) = 0,$$

which has three real roots $a_1 = 0$, $a_2 = \frac{1}{3}$, $a_3 = -\frac{1}{3}$. They correspond with the quintuple of real numbers with the same components, which are the solution of the given system of equations.

Conclusion. The given system of equations has exactly three solutions in real numbers. These are the triples $(a, b, c, d, e) = (\ell, \ell, \ell, \ell, \ell)$, where $\ell \in \{0, \frac{1}{3}, -\frac{1}{3}\}$.

Problem 21 ([16], problem 25)

Solve the system of equations in real numbers

$$\begin{aligned} \frac{x^2}{y} + \frac{y^2}{x} &= 12, \\ \frac{1}{x} + \frac{1}{y} &= \frac{1}{3}. \end{aligned}$$

Solution. Firstly, let us realize that $x \neq 0 \neq y$. If we eliminate the fractions from both equations, we rewrite the given system of equations in the form

$$(x + y)[(x + y)^2 - 3xy] = 12xy, \tag{1}$$

$$3(x + y) = xy. \tag{2}$$

If we use *substitution* $u = x + y$ and $v = xy$ and eliminate the unknown v, we get (after simple manipulation) the following cubic equation

$$u(u^2 - 9u - 36) = 0, \tag{3}$$

which has three real roots $u_1 = 0$, $u_2 = 12$ and $u_3 = -3$. We can easily see that the root $u_1 = 0$ contradicts the conditions of the problem. Hence the equation (3) implies that either $u = 12$ or $u = -3$.

- If $u = 12$, we approach to solve the system of equations

  $$x + y = 12, \qquad xy = 36,$$

  which means that $x = y = 6$.

- If $u = -3$, we get the system of equations

  $$x + y = -3, \qquad xy = -9.$$

  The given system of equations has exactly two solutions in real numbers, i.e. $\left(\frac{3}{2}(-1 \pm \sqrt{5}); \frac{3}{2}(-1 \mp \sqrt{5})\right)$.

Conclusion. The given system of equations has exactly three real solutions, i.e. $(x, y) \in \{(6; 6), \left(\frac{3}{2}(-1 + \sqrt{5}); \frac{3}{2}(-1 - \sqrt{5})\right), \left(\frac{3}{2}(-1 - \sqrt{5}); \frac{3}{2}(-1 + \sqrt{5})\right)\}$. We do not need to check the solution.

Problem 22 (8th CPS Junior MO, 2019, T–4, J. Švrček)

Determine all values of the expression $V = xy + yz + zx$ with real numbers x, y, z satisfying the following conditions

$$x^2 - yz = y^2 - zx = z^2 - xy = 2.$$

Solution. Firstly, we can rewrite the given conditions in the form

$$\begin{aligned} x^2 - yz &= 2, && (1)\\ y^2 - zx &= 2, && (2)\\ z^2 - xy &= 2 && (3) \end{aligned}$$

and equivalently – after subtracting (1) – (2) and (1) – (3)

$$\begin{aligned} x^2 - yz &= 2,\\ (x^2 - y^2) + (x - y)z = (x - y)(x + y + z) &= 0,\\ (x^2 - z^2) + (x - z)y = (x - z)(x + y + z) &= 0. \end{aligned}$$

Since the case $x = y = z$ is impossible, the equality $x + y + z = 0$ (in each another case) must hold. Adding up (1)-(3) we further obtain

$$x^2 + y^2 + z^2 - (xy + yz + zx) = (x + y + z)^2 - 3(xy + yz + zx) = 6.$$

Substituting $x + y + z = 0$ and after easy calculation we immediately get

$$xy + yz + zx = -2.$$

Conclusion. The expression V takes under the given conditions the value -2, only (always).

Note. It is easy to show that a set of all triples (x, y, z) of real numbers satisfying the given conditions is *non–empty.*

Problem 23 (10^{th} CPS Junior MO, 2021, I–2, J. Švrček)
In the domain of integers solve the following system of equations

$$\begin{aligned} x^2 &= yz+1, \\ y^2 &= zx+1, \\ z^2 &= xy+1. \end{aligned}$$

Solution. We can rewrite the given system of equations in the form

$$\begin{aligned} x^2-1 &= yz, && (1) \\ y^2-1 &= zx, && (2) \\ z^2-1 &= xy. && (3) \end{aligned}$$

Multiplying (1)–(3) we obtain

$$(x^2-1)(y^2-1)(z^2-1) = x^2y^2z^2. \tag{4}$$

If $xyz \neq 0$, we from (4) further get

$$\left(1-\frac{1}{x^2}\right)\left(1-\frac{1}{y^2}\right)\left(1-\frac{1}{z^2}\right) = 1. \tag{5}$$

Since $0 \le 1 - 1/x^2 < 1$, $0 \le 1 - 1/y^2 < 1$ and $0 \le 1 - 1/z^2 < 1$, we can see that the fulfilling of (5) in this case is *impossible.* Thus, (at least) one of the unknowns x, y, z must be equal 0.

If $x = 0$ then $y = 1$, $z = -1$ or $y = -1$, $z = 1$ and similarly for $y = 0$ and $z = 0$.

Conclusion. The given system of equations has exactly six solutions, namely:
$(x, y, z) \in \{(0; 1; -1), (0; -1; 1), (1; 0; -1), (-1; 0; 1), (1; -1; 0), (-1; 1; 0)\}$.

Problem 24 (Salvadorean MO, 2019, 10^{th} Grade, modified)
Find all triples (a, b, c) of integers satisfying the following system of equations

$$\begin{aligned} a^3+a^2+b^2 &= 0, && (1) \\ b^3+b^2+c^2 &= 0, && (2) \\ c^3+c^2+a^2 &= 0. && (3) \end{aligned}$$

Solution. If some of unknowns (let's say a) is equal 0, then also other two unknowns b, c are equal 0. Thus, we have the solution $(a, b, c) = (0; 0; 0)$.

Further, let $abc \neq 0$. Rewriting (1) in the form $a^2(a+1) = -b^2 < 0$ we get $a < -1$, i.e. $a \leq -2$. Similarly, from (2) and (3) we get $b \leq -2$ and $c \leq -2$.

Adding up all equations (1)–(3) we obtain (after easy manipulation)

$$a^2(a+2) + b^2(b+2) + c^2(c+2) = 0. \tag{4}$$

Since $(a \leq -2) \wedge (b \leq -2) \wedge (c \leq -2)$ and a^2, b^2, c^2 are positive integers, we see that there exists unique solution of (4) which is the triple $(a, b, c) = (-2; -2; -2)$.

Conclusion. After checking (it is necessary for this part of solving) we can see that the given system of equations has exactly two solutions, namely $(a, b, c) = = (-2; -2; -2)$ and $(a, b, c) = (0; 0; 0)$.

Problem 25 (69th Belorussian MO, 2019)

Determine all triples (x, y, z) of real numbers satisfying the following system of equations

$$(x+1)(x^2+1) = y^3 + 1, \tag{1}$$
$$(y+1)(y^2+1) = z^3 + 1, \tag{2}$$
$$(z+1)(z^2+1) = x^3 + 1. \tag{3}$$

Solution. It is easy to see that the triples $(0; 0; 0)$, $(-1; -1; -1)$ of real numbers satisfy the given system of equations. We will show that there is no other real solution. The system of equations (1)–(3) yield that if $x = -1$ then from (1) it follows $y = -1$ and from (2) we subsequently get $z = -1$. Similarly, if e.g. $x = 0$ then $y = z = 0$.
Further, suppose that none of x, y and z equals -1 or 0. Multiplying (1)–(3) by the same sides we obtain

$$(x+1)(x^2+1)(y+1)(y^2+1)(z+1)(z^2+1) = (y^3+1)(z^3+1)(y^3+1) =$$
$$= (y+1)(y^2-y+1)(z+1)(z^2-z+1)(x+1)(x^2-x+1).$$

Reducing by $(x+1)(y+1)(z+1) \neq 0$, we get

$$(x^2+1)(y^2+1)(z^2+1) = (x^2-x+1)(y^2-y+1)(z^2-z+1). \tag{4}$$

Note that if $x < -1$ then, according to (1), $y < -1$ and, according to (2), $z < -1$. In this case all factors in the right side (4) are greater than corresponding factors on the left side of (4) which is a contradiction.

Now, subtracting 1 in left and right sides of all three equations (1)–(3) and after easy calculation we rewrite the given system (1)–(3) in the form

$$x(x^2 + x + 1) = y^3, \tag{5}$$

$$y(y^2 + y + 1) = z^3, \tag{6}$$

$$z(z^2 + z + 1) = x^3. \tag{7}$$

Note that if $x > 0$ then, according to (5) and (6) we have $y > 0$ and $z > 0$. In this case, the product of all factors on the right side of (4) is smaller than the product of all factors on the left side of (4), which is again a contradiction.
It remains to consider the case, if all real numbers x, y, z belong to the interval $(-1; 0)$ and, in particular, are negative. Again, all factors on the right side of (4) are greater than the corresponding factors on the left side. Thus, there are no other real solution of the given system (1)–(3).

CONCLUSION. Exactly two triples (x, y, z) of real numbers satisfy the given system of equations, namely $(0; 0; 0)$ and $(-1; -1; -1)$.

4 Unsolved problems

Problem 26 (25th MO, 1975/1976, B–II–3b)

Solve the system of equations in the positive real numbers

$$\begin{aligned} x_1 + \frac{1}{x_2} &= 2, \\ x_2 + \frac{1}{x_3} &= 2, \\ x_3 + \frac{1}{x_1} &= 2. \end{aligned}$$

[Solution: $(x_1, x_2, x_3) = (1; 1; 1)$.]

Problem 27

Solve the system of equations in real numbers

$$\begin{aligned} x + 2y + 3z &= 28, \\ x^2 + y^2 + z^2 &= 56. \end{aligned}$$

[Solution: $(x, y, z) = (2; 4; 6)$.]

Problem 28 (32nd British MO, 1996)

Solve the system of equations in the positive real numbers

$$\begin{aligned} a + b + c + d &= 12, \\ abcd &= 27 + ab + ac + ad + bc + bd + cd. \end{aligned}$$

[Solution: $(a, b, c, d) = (3; 3; 3; 3)$.]

Problem 29 (Swedish MO, 1989)

Solve the system of equations in the positive real numbers

$$\begin{aligned} x + y^2 + z^3 &= 3, \\ y + z^2 + x^3 &= 3, \\ z + x^2 + y^3 &= 3. \end{aligned}$$

[Solution: $(x, y, z) = (1; 1; 1)$.]

Problem 30 (31st MO, 1981/1982, A–II–2)

Determine all n–tuples $(x_1, x_2, \dots, x_n)$ of positive real numbers satisfying the system of equations

$$\begin{aligned} x_1 + x_2 + \ldots + x_n &= \frac{1}{4}, \\ \frac{1}{x_1} + \frac{4}{x_2} + \ldots + \frac{n^2}{x_n} &= n^2(n+1)^2. \end{aligned}$$

$\left[\text{Solution: } x_i = \dfrac{i}{2n(n+1)}, \text{ where } i = 1, 2, \dots, n.\right]$

Problem 31 (57th MO, 2007/2008, A–III–1, J. Švrček)

Solve the system of equations in real numbers

$$\begin{aligned} x + y^2 &= y^3, \\ y + x^2 &= x^3. \end{aligned}$$

[Solution: $(x, y) \in \{(0; 0), \left(\frac{1}{2}(1+\sqrt{5}); \frac{1}{2}(1+\sqrt{5})\right), \left(\frac{1}{2}(1-\sqrt{5}); \frac{1}{2}(1-\sqrt{5})\right)\}$.]

Problem 32 (61st MO, 2011/2012, A–S–1, P. Calábek)

Solve the system of equations in real numbers

$$\begin{aligned} x + 3y &= 4y^3, \\ y + 3x &= 4x^3. \end{aligned}$$

[There are nine solutions to the system of equations. These are pairs (x, y):

$$\begin{gathered} (0; 0),\ (1; 1),\ (-1; -1),\ (\tfrac{1}{2}\sqrt{2}; -\tfrac{1}{2}\sqrt{2}),\ (-\tfrac{1}{2}\sqrt{2}; \tfrac{1}{2}\sqrt{2}), \\ (\tfrac{1}{4} + \tfrac{1}{4}\sqrt{5}; \tfrac{1}{4} - \tfrac{1}{4}\sqrt{5}),\ (\tfrac{1}{4} - \tfrac{1}{4}\sqrt{5}; \tfrac{1}{4} + \tfrac{1}{4}\sqrt{5}), \\ (-\tfrac{1}{4} + \tfrac{1}{4}\sqrt{5}; -\tfrac{1}{4} - \tfrac{1}{4}\sqrt{5}),\ (-\tfrac{1}{4} - \tfrac{1}{4}\sqrt{5}; -\tfrac{1}{4} + \tfrac{1}{4}\sqrt{5}).\,] \end{gathered}$$

Problem 33 (62nd OM, Poland, 2010/2011, II–1)

Solve the system of equations in real numbers

$$\begin{aligned} (x - y)(x^3 + y^3) &= 7, \\ (x + y)(x^3 - y^3) &= 3. \end{aligned}$$

$\left[\text{Solution: } (x, y) = \left(\frac{2}{\sqrt[4]{3}}; -\frac{1}{\sqrt[4]{3}}\right) \text{ a } (x, y) = \left(-\frac{2}{\sqrt[4]{3}}; \frac{1}{\sqrt[4]{3}}\right)\right].$

Problem 34 (63rd OM[2], Poland, 2011/2012, I–1)

Solve the system of equations in real numbers

$$\begin{aligned}(x+y)^3 &= 8z,\\ (y+z)^3 &= 8x,\\ (z+x)^3 &= 8y.\end{aligned}$$

[Solution: $(x,y,z) \in \{(0;0;0),\ (1;1;1),\ (-1;-1;-1)\}$.]

Problem 35 (63rd OM, Poland, 2011/2012, II–1)

Solve the system of equations in real numbers

$$\begin{aligned}a^3+b &= c,\\ b^3+c &= d,\\ c^3+d &= a,\\ d^3+a &= b.\end{aligned}$$

[Solution: $(a,b,c,d) \in \{(0;0;0;0),\ (\sqrt{2};-\sqrt{2};\sqrt{2};-\sqrt{2}),\ (-\sqrt{2};\sqrt{2};-\sqrt{2};\sqrt{2})\}$.]

Problem 36 (19th OJM[3], 1979/1980)

Solve the system of equations in real numbers

$$\begin{aligned}2x+x^2y &= y,\\ 2y+y^2z &= z,\\ 2z+z^2x &= x.\end{aligned}$$

[Solution: $(x,y,z) = (\operatorname{tg}\frac{k\pi}{7};\operatorname{tg}\frac{2k\pi}{7};\operatorname{tg}\frac{4k\pi}{7})$ for $k=-3,-2,-1,0,1,2,3$, see [22].]

Problem 37 (19th Mathematical Duel, 2011, J. Švrček, [6])

Solve the system of equations in real numbers

$$\begin{aligned}x^4+1 &= 2yz,\\ y^4+1 &= 2zx,\\ z^4+1 &= 2xy.\end{aligned}$$

[Solution: $(x,y,z) \in \{(1;1;1),(-1;-1;-1)\}$.]

---

[2] Olimpiada Matematyczna

[3] Olympiade junger Mathematiker der DDR

Problem 38 (59th OM, Poland, 2007/2008, I–1)

Solve the system of equations in real numbers

$$\begin{aligned} x^5 &= 5y^3 - 4z, \\ y^5 &= 5z^3 - 4x, \\ z^5 &= 5x^3 - 4y. \end{aligned}$$

[The system of equations has 5 solutions: $(x, y, z) \in \{(-2; -2; -2), (-1; -1; -1), (0; 0; 0), (1; 1; 1), (2; 2; 2)\}$.]

Problem 39 (20th Mathematical Duel, 2012, J. Uryga)

Solve the system of equations in integers

$$\begin{aligned} x + \frac{2}{y} &= z, \\ y + \frac{4}{z} &= x, \\ z - \frac{6}{x} &= y. \end{aligned}$$

[The system of equations has 8 solutions: $(x, y, z) \in \{(-3; 1; -1), (2; 1; 4), (-2; 2; -1), (3; 2; 4), (3; -1; 1), (-2; -1; -4), (2; -2; 1), (-3; -2; -4)\}$.]

Problem 40 (16th Mathematical competition Poland – Austria, 1992/1993)

Solve the system of equations in real numbers

$$\begin{aligned} x^3 + y &= 3x + 4, \\ 2y^3 + z &= 6y + 6, \\ 3z^3 + x &= 9z + 8. \end{aligned}$$

[Solution: $(x, y, z) = (2; 2; 2)$.]

Problem 41 (14th Mathematical competition Poland – Austria, 1990/1991)

Solve the system of equations in real numbers

$$\begin{aligned} (x^2 - 6x + 13)y &= 20, \\ (y^2 - 6y + 13)z &= 20, \\ (z^2 - 6z + 13)x &= 20. \end{aligned}$$

[Solution: $(x, y, z) = (4; 4; 4)$.]

Problem 42 (8^{th} Austrian MO[4], 1977, [1])

Let n be a positive integer ($n \geq 2$). Solve the system of equations in real numbers

$$\begin{aligned} x_1(x_1-1) &= x_2-1, \\ x_2(x_2-1) &= x_3-1, \\ &\dots \\ x_{n-1}(x_{n-1}-1) &= x_n-1, \\ x_n(x_n-1) &= x_1-1. \end{aligned}$$

[Solution: $(x_1, x_2, \ldots, x_n) = (1; 1; \ldots; 1)$.]

Problem 43 (Russian MO, 1992)

Solve the system of equations in real numbers

$$\begin{aligned} (1+x)(1+x^2)(1+x^4) &= 1+y^7, \\ (1+y)(1+y^2)(1+y^4) &= 1+x^7. \end{aligned}$$

[Solution: $(x, y) \in \{(0; 0), (-1; -1)\}$.]

Problem 44 (52^{nd} MO, 2002/2003, A–II–3, J. Švrček)

Solve the system of equations in real numbers

$$\begin{aligned} \log_x(y+z) &= 2, \\ \log_y(z+x) &= 2, \\ \log_z(z+y) &= 2. \end{aligned}$$

[Solution: $(x, y, z) = (2; 2; 2)$.]

Problem 45 (10^{th} Austrian MO, 1979, [1])

Solve the system of equations in real numbers

$$\begin{aligned} x(y+z) &= 35, \\ y(z+x) &= 32, \\ z(x+y) &= 27. \end{aligned}$$

[Solution: $(x, y, z) \in \{(5; 4; 3), (-5; -4; -3)\}$.]

---

[4] Österreichische Mathematik Olympiade

Problem 46 (Vietnamese MO, 1994)

Solve the system of equations in real numbers

$$\begin{aligned} x^2 + 3x + \log(2x+1) &= y, \\ y^2 + 3y + \log(2y+1) &= x. \end{aligned}$$

[Solution: $(x, y) = (0; 0)$.]

Problem 47

Solve the system of equations in real numbers

$$\begin{aligned} x^2 + y^2 + z^2 &= 3, \\ xy + yz + zx &= 3. \end{aligned}$$

[Solution: $(x, y, z) \in \{(1; 1; 1), (-1; -1; -1)\}$.]

Problem 48 ([16], problem 42)

Solve the system of equations in real numbers

$$\begin{aligned} x + y + z &= 2, \\ 2xy - z^2 &= 4. \end{aligned}$$

[Solution: $(x, y, z) = (2; 2; -2)$.]

Problem 49 (Vietnamese MO, 2004)

Solve the system of equations in real numbers

$$\begin{aligned} x^3 + 3xy^2 &= -49, \\ y^2 - 8xy + y^2 &= 8y - 17x. \end{aligned}$$

[Solution: $(x, y) \in \{(-1; 4), (-1; -4)\}$.]

Problem 50 (Vietnamese MO, 2004)

Solve the system of equations in real numbers

$$\begin{aligned} x^3 + x(y-z)^2 &= \ \ 2, \\ y^3 + y(z-x)^2 &= 30, \\ z^3 + z(x-y)^2 &= 16. \end{aligned}$$

[Solution: $(x, y, z) = (1; 3; 2)$.]

Problem 51 ([13], problem 19)

Solve the system of equations in real numbers

$$\begin{aligned}(x+y)^2 &= z,\\ (y+z)^2 &= x,\\ (z+x)^2 &= y.\end{aligned}$$

[Solution: $(x, y, z) \in \{(0; 0; 0), (\frac{1}{4}; \frac{1}{4}; \frac{1}{4})\}$.]

Problem 52 (Vietnamese MO, 2006)

Solve the system of equations in real numbers

$$\begin{aligned}x^3+3x^2+2x-5 &= y,\\ y^3+3y^2+2y-5 &= z,\\ z^3+3z^2+2z-5 &= x.\end{aligned}$$

[Solution: $(x, y, z) = (1; 1; 1)$.]

Problem 53

Solve the system of equations in real numbers

$$\begin{aligned}x^3 &= y+2y^3,\\ y^3 &= z+2z^3,\\ z^3 &= x+2x^3.\end{aligned}$$

[Solution: $(x, y, z) = (0; 0; 0)$.]

Problem 54 ([13], problem 15)

Solve the system of equations in real numbers

$$\begin{aligned}|x_1-x_2| &= \frac{\sqrt{5}}{2}x_3,\\ |x_2-x_3| &= \frac{\sqrt{5}}{2}x_1,\\ |x_3-x_1| &= \frac{\sqrt{5}}{2}x_2.\end{aligned}$$

Help: use extremal principle.

[Solution: $(x_1, x_2, x_3) = (0; 0; 0)$.]

Problem 55 (CPS competition before 51^{st} IMO, 2010)

Solve the system of equations in the positive real numbers

$$\begin{aligned} a\sqrt{b} - c &= a, \\ b\sqrt{c} - a &= b, \\ c\sqrt{a} - b &= c. \end{aligned}$$

[Solution: $(a, b, c) = (4; 4; 4)$.]

Problem 56

Solve the system of equations in real numbers

$$\begin{aligned} x^3 + y^3 &= 5, \\ x^2y + xy^2 &= 1. \end{aligned}$$

Help: use substitution $u = x + y$, $v = xy$.

[Solution: $(x, y) \in \left\{ \left(\frac{1}{2}(2+\sqrt{2}); \frac{1}{2}(2-\sqrt{2})\right), \left(\frac{1}{2}(2-\sqrt{2}); \frac{1}{2}(2+\sqrt{2})\right) \right\}$.]

Problem 57 (10^{th} MEMO, 2016, Problem T–1)

Solve the system of equations in real numbers

$$\begin{aligned} a^2 + ab + c &= 0, \\ b^2 + bc + a &= 0, \\ c^2 + ca + b &= 0. \end{aligned}$$

[Solution: $(a, b, c) \in \{(0; 0; 0), (-\frac{1}{2}; -\frac{1}{2}; -\frac{1}{2})\}$.]

Problem 58 (72^{nd} MO, 2022/2023, A–I–1, J. Švrček)

In the domain of real numbers solve the system of equations

$$\begin{aligned} 2x + \lfloor y \rfloor &= 2022, \\ 3y + \lfloor 2x \rfloor &= 2023. \end{aligned}$$

(The symbol $\lfloor a \rfloor$ denotes *the lower integer part* of a real number a.)

[Solution: $(x, y) = (1011; \frac{1}{3})$.]

Problem 59 (J. Švrček)
In real numbers solve the system of equations

$$\begin{aligned} 3x + \lfloor y \rfloor &= 10, \\ \lfloor 4x \rfloor + x + y &= 17. \end{aligned}$$

[Solution: $(x, y) \in \{(\frac{10}{3}; \frac{2}{3}), (\frac{11}{3}; -\frac{2}{3})\}$.]

Problem 60 (Matematyczna Liga zadaniowa, Poland, 2018/2019, [12])
Solve the system of equations in real numbers

$$\begin{aligned} (x+y)^2 + 2x + 2y &= z + 1, \\ (y+z)^2 + 2y + 2z &= x + 1, \\ (z+x)^2 + 2z + 2x &= y + 1. \end{aligned}$$

[Solution: $(x, y, z) \in \{(-1, -1, -1), (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}),$
$\left(\frac{1}{8}(-3 + 3\sqrt{17}), \frac{1}{8}(-7 - \sqrt{17}), \frac{1}{8}(-7 - \sqrt{17})\right),$
$\left(\frac{1}{8}(-7 - \sqrt{17}), \frac{1}{8}(-3 + 3\sqrt{17}), \frac{1}{8}(-7 - \sqrt{17})\right),$
$\left(\frac{1}{8}(-7 - \sqrt{17}), \frac{1}{8}(-7 - \sqrt{17}), \frac{1}{8}(-3 + 3\sqrt{17})\right),$
$\left(\frac{1}{8}(-3 - 3\sqrt{17}), \frac{1}{8}(-7 + \sqrt{17}), \frac{1}{8}(-7 + \sqrt{17})\right),$
$\left(\frac{1}{8}(-7 + \sqrt{17}), \frac{1}{8}(-3 - 3\sqrt{17}), \frac{1}{8}(-7 + \sqrt{17})\right),$
$\left(\frac{1}{8}(-7 + \sqrt{17}), \frac{1}{8}(-7 + \sqrt{17}), \frac{1}{8}(-3 - 3\sqrt{17})\right)\}$.]

References

[1] Baron, G. – Windischbacher, E.: Österreichische Mathematik Olympiaden 1970–1989, Universitätverlag Wagner, Innsbruck, 1990.

[2] Berinde, V.: Exploring, Investigating and Discovering in Mathematics. Birkäuser Verlag, Basel–Boston–Berlin 2004.

[3] Calda, E.: Rovnice ve škole neřešené Praxe učitele matematiky, fyziky a informatiky. Prometheus, Praha 1995.

[4] Calda, E.: Středoškolská matematika pod mikroskopem, sbírka řešených úloh. Prometheus, Praha 2006.

[5] Engel, A.: Problem-Solving Strategies. Springer-Verlag, New York–Berlin–Heidelberg 1998.

[6] Geretschläger, R. – Kalinowski, J. – Švrček, J.: A Central European Olympiad – The Mathematical Duel. World Scientific Publishing, Singapore 2018.

[7] Charvát, J. – Zhouf, J. – Boček, L.: Matematika pro gymnázia (rovnice a nerovnice). Prometheus, Praha 1999.

[8] Chau, L. H. – Khoi, L. H.: Selected Problems of the Vietnamese Mathematical Olympiad. World Scientific Publishing, Singapore 2010.

[9] Herman, J. – Kučera, R. – Šimša, J.: Metody řešení matematických úloh I. Masarykova Univerzita, Brno 1996 (2. vydání).

[10] Kanel'-Belov, A. J. – Koval'dži, A. K.: Kak rešajut nestandartnyje zadači (rusky). MCNMO, Moskva 1997.

[11] Klamkin, M. S.: USA Mathematical Olympiads 1972–1986. The MAA, Washington D.C., 1988.

[12] Klemens, W. – Meissner, L. – Mosćicka, M. – Ormaniec, M. – Szymczyk, T.: Matematyczna Liga zadaniowa. Wyd. Szkolne Omega, Kraków 2019.

[13] Kurlyandchik, L.: Matematyka elementarna w zadanich 1. Aksjomat, Toruń, 2005.

[14] Kuřina, F.: Matematika a řešení úloh. Jihočeská univerzita v Českých Budějovicích, Pedagogická fakulta, České Budějovice 2011.

[15] Larson, L. C.: Problem Solving Through Problems. Springer-Verlag, New York–Berlin–Heidelberg 1983.

[16] Lidskij, V. B. a kol. Úlohy z elementární matematiky. SPN, Praha 1965.

[17] Odvárko, O. – Calda, E. – Šedivý, J. – Židek, S.: Metody řešení matematických úloh. SPN, Praha 1990.

[18] Di Pasquale, E. – Do, N. – Mathews, D.: Problem Solving Tactics. Australian Mathematics Trust, Canberra 2014.

[19] Pólya, G.: How to Solve It. Doubleday, second edition, 1957.

[20] Soifer, A.: Mathematics as Problem Solving (Second Edition). Springer–Verlag, New York–Berlin–Heidelberg 2009.

[21] Švrček, J. – Calábek, P.: Sbírka netradičních matematických úloh. Prometheus, Praha 2007.

[22] Švrček, J.: Goniometrické substituce. Matematika–fyzika–informatika, Vol. 10, 2000/2001, No. 10.

[23] Švrček, J.: The Systems of Cyclic Equations. Mathematics and Informatics Quarterly, Vol. 11, 2001, No. 1.

[24] Švrček, J.: On Some Type of Systems of Cyclic Equations in Czech and Slovak Maths Competitions. Canberra: Mathematics Competitions, Australian Mathematics Trust Publishing, 2003, Vol. 16, No. 2.

[25] Tabov, J. B. – Taylor, P. J.: Methods of Problem Solving (Book 1). Australian Mathematics Trust, Canberra 1996.

[26] Tabov, J. B. – Taylor, P. J.: Methods of Problem Solving (Book 2). Australian Mathematics Trust, Canberra 2002.

[27] Tabov, J. B. – Kolev, E. M. – Taylor, P. J.: Methods of Problem Solving (Book 3). Australian Mathematics Trust, Canberra 2012.

[28] Tao, T.: Solving Mathematical Problems. Oxford University Press, New York 2006

[29] Wickelgren, W. A.: How to Solve Mathematical Problems. Dover Publishing, New York 1974.

[30] Zeitz, P.: The Art and Craft of Problem Solving. John Wiley & Sons, Inc., 1999.